

# NON-UNIFORM HYPERBOLICITY AND EXISTENCE OF ABSOLUTELY CONTINUOUS INVARIANT MEASURES

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**ABSTRACT.** We prove that for certain partially hyperbolic skew-products, non-uniform hyperbolicity along the leaves implies existence of a finite number of ergodic absolutely continuous invariant probability measures which describe the asymptotics of almost every point. The main technical tool is an extension for sequences of maps of a result of de Melo and van Strien relating hyperbolicity to recurrence properties of orbits. As a consequence of our main result, we also obtain a partial extension of Keller's theorem guaranteeing the existence of absolutely continuous invariant measures for non-uniformly hyperbolic one dimensional maps.

## 1. INTRODUCTION

In this paper we study the existence of absolutely continuous invariant probability measures for non-uniformly expanding maps in dimensions larger than 1.

It is a classical fact (see Mañé, [13]) that every uniformly expanding smooth map on a compact manifold admits a unique ergodic absolutely continuous invariant measure, and this measure describes the asymptotics of almost every point. Moreover, see Bowen [6], uniformly hyperbolic diffeomorphisms also have a finite number of such *physical measures*, describing the asymptotics of almost every point. Actually, in this case, the physical measures are absolutely continuous only along certain directions, namely, the expanding ones.

The present work is motivated by the question of knowing, to what extent, weaker forms of hyperbolicity are still sufficient for the existence of such measures. A precise statement in this direction is:

**Conjecture** (Viana, [23]). *If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some physical measure.*

Two main results provide some evidence in favor of this conjecture. The older one is the remarkable theorem of Keller [11] stating that *for maps of the interval with finitely many critical points and non-positive Schwarzian derivative, existence of absolutely continuous invariant probability is guaranteed by positive Lyapunov exponents, i.e.,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| > 0 \quad (1.1)$$

*on a positive Lebesgue measure set of points  $x$*  (see Subsection 3.1 for definitions involved). In fact, Keller proved the existence of a finite number of these measures whose union of basins have full Lebesgue measure, in the case that (1.1) holds for Lebesgue almost every point.

Then, more recently, Alves, Bonatti and Viana [4] proved that *every non-uniformly expanding local diffeomorphism on any compact manifold admits a finite number of ergodic absolutely continuous invariant*

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measures describing the asymptotics of almost every point. This notion of non-uniform expansion means that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\|^{-1} \geq c > 0 \quad (1.2)$$

almost everywhere. Alves, Bonatti and Viana [4] also give a version of this result for maps with singularities, that is, which fail to be a local diffeomorphism on some subset  $\mathcal{S}$  of the ambient manifold. However, due to the presence of singularities they need an additional hypothesis (of slow recurrence to the singular set  $\mathcal{S}$ ) which is often difficult to verify. Given that Keller's theorem has no hypothesis about the recurrence to the singular set (in his case  $\mathcal{S} = \{\text{critical points}\}$ ), one may ask to what extent this condition is really necessary.

This question was the starting point of the present work. Before giving our statements, let us mention a few related results.

One partial extension of both Keller [11] and Alves, Bonatti and Viana [4], was obtained recently by Pinheiro [16]: he keeps the slow recurrence condition but is able to weaken the hyperbolicity condition substantially, replacing  $\liminf$  by  $\limsup$  in (1.2).

Another important result was due to Tsujii [21]:  $C^r$  generic partially hyperbolic endomorphisms on a compact surface admit finitely many ergodic physical measures and the union of their basins is a total Lebesgue measure set. When the center Lyapunov exponents are positive, these measures are absolutely continuous.

Our own results holds for a whole, explicitly defined, family of transformations on surfaces. We prove existence and finiteness of ergodic absolutely continuous invariant measures, assuming only non-uniform expansion (slow recurrence is not necessary).

Motivated by a family of maps introduced by Viana [22] and studied by several other authors (see for example [2, 5, 8, 19, 3]) we consider transformations of the form  $\varphi : \mathbb{T}^1 \times I_0 \rightarrow \mathbb{T}^1 \times I_0$ ,  $(\theta, x) \mapsto (g(\theta), f(\theta, x))$ , where  $g$  is a uniformly expanding circle map, each  $f(\theta, \cdot)$  is a smooth interval map with non-positive Schwarzian derivative, and  $\varphi$  is partially hyperbolic with vertical central direction:

$$|\partial_\theta g(\theta)| > |\partial_x f(\theta, x)| \quad \text{at all points.}$$

We prove that if  $\varphi$  is non-uniformly expanding then it admits some absolutely continuous invariant probability. Moreover, there exist finitely many ergodic absolutely continuous invariant probabilities whose union of basins is a full Lebesgue measure set.

The Viana maps [22] correspond to the case when  $g$  is affine,  $g(\theta) = d\theta \pmod{1}$  with  $d \gg 1$ , and  $f$  has the form  $f(\theta, x) = a_0 + \alpha \sin(2\pi\theta) - x^2$  (actually, [22] deals also with arbitrary small perturbations of such maps). It was shown in [22] that Viana maps are indeed non-uniformly expanding. Moreover, Alves [2] proved that they have a unique physical measure, which is absolutely continuous and ergodic. Their methods hold even for a whole open set of maps not necessarily of skew-product form. In fact, the argument of [2] rely on a proof of slow recurrence to the critical set which in that case is the circle  $\mathbb{T}^1 \times \{0\}$ .

For the family of maps which we consider (see Theorem A), we do not assume the slow recurrence condition, fundamental in [4], [2] and [16]. On the other hand, our method is completely different from the one used in the mentioned works. We view  $\varphi$  as a family of smooth maps of the interval, namely, its restrictions to the vertical fibers  $\{\theta\} \times I_0$ . Thus, our main technical tool is an extension for such families of maps of a result proved by de Melo and van Strien [14, Theorem V.3.2, page 371] for individual unimodal maps saying, in a few words, that positive Lyapunov exponents manifest themselves at a macroscopic level: intervals that are mapped diffeomorphically onto large domains under iterates of the map. This, in turn, allows us to make use of the hyperbolic times technique similar to the one introduced by Alves, Bonatti and Viana [4].

Let us remark that in the setting of piecewise expanding maps in high dimensions, there are several works which deal with existence of absolutely continuous invariant measures. Among them, let us mention [1, 7, 9, 10, 18]. In all the cases, additional conditions on the expanding constants and (or) the boundary behavior are required.

**1.1. Organization of the paper.** This paper is organized as follows. In Section 2 we give the precise statement of the main results. In section 3 we introduce a few preliminary facts, which will be useful in the sequel. In section 4 we prove our Theorem B, which is the extension of [14, Theorem V.3.2, page 371] mentioned before. The section 5 contains the proof of one partial extension of Keller's theorem.

In section 6 we prove another key result (Proposition 6.3): for each interval which is mapped diffeomorphically onto a large domain under an iterate of the skew-product, there exists an open set containing this interval which is sent diffeomorphically onto its image under the same iterate. Moreover, this map has bounded distortion and the measure of the image is bounded away from zero. We call these iterates *hyperbolic-like times*, because their behavior is similar to hyperbolic times introduced in [4].

In section 7 we combine the main lemma (Lemma 4.1) used in the proof of Theorem B, with the Pliss Lemma to conclude that the set of points with infinitely many (and even positive density of) hyperbolic-like times has positive Lebesgue measure. The construction of the absolutely continuous invariant measure for the skew-product  $\varphi$  follows along well-known lines, as we explain in subsection 7.3. Finally, on subsection 7.4, we prove the ergodicity of the measure and the existence of finitely many SRB measures.

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## 2. STATEMENT OF THE RESULTS

Let us present the precise statements of our results.

**2.1. Non-uniformly expanding skew-products.** Let  $I_0$  be an interval and let  $\mathbb{T}^1$  be the circle. We consider  $C^3$  partially hyperbolic skew-products defined on  $\mathbb{T}^1 \times I_0$ , with critical points in the vertical direction. The mappings we consider are precisely

$$\begin{aligned} \varphi : \mathbb{T}^1 \times I_0 &\rightarrow \mathbb{T}^1 \times I_0 \\ (\theta, x) &\rightarrow (g(\theta), f(\theta, x)) \end{aligned}$$

where  $g$  is a uniformly expanding smooth map on  $\mathbb{T}^1$  and  $f_\theta : I_0 \rightarrow I_0$ ,  $x \rightarrow f(\theta, x)$  is a smooth map, possibly with critical points, for every  $\theta \in \mathbb{T}^1$ . We assume our map is partially hyperbolic, it means that satisfies (3.1) below (see Subsection 3.2).

In the result of Alves, Bonatti and Viana (see [4, Theorem C]), the set  $S$  of singular points of  $\varphi$  satisfies the non-degenerate singular set conditions. These conditions allow the co-existence of critical points and points with  $|\det D\varphi| = \infty$ . We will only admit critical points.

We denote by  $\mathcal{C}$  the set of critical points of  $\varphi$  and by  $\mathcal{C}_\theta$  the set of critical points contained in the  $\theta$ -vertical leaf. By  $\text{dist}_{\text{vert}}$  we denote the distance induced by the Riemmanian metric in the vertical leaf, i.e, if  $z = (\theta, x)$  for some  $x$ ,  $\text{dist}_{\text{vert}}(z, \mathcal{C}) = \text{dist}(z, \mathcal{C}_\theta)$ .

Let  $M = \mathbb{T}^1 \times I_0$  and  $\mathcal{C} \subset M$  a compact set. We consider a  $C^3$  skew product map  $\varphi : M \rightarrow M$  which is a local  $C^3$  diffeomorphism in the whole manifold except in a critical set  $\mathcal{C}$  such that:

$$(F_1) \quad p = \sup \# \mathcal{C}_\theta < \infty ;$$

there exists  $B > 0$  such that, for every  $z \in M \setminus \mathcal{C}$ ,  $w \in M$  with  $\text{dist}(z, w) < \text{dist}_{\text{vert}}(z, \mathcal{C})/2$

$$(F_2) \quad \left| \log |\partial_x f(z)| - \log |\partial_x f(w)| \right| \leq \frac{B}{\text{dist}_{\text{vert}}(z, \mathcal{C})} \text{dist}(z, w);$$

and for all  $\theta \in \mathbb{T}^1$

$$(F_3) \quad Sf(\theta, x) \leq 0, \text{ for } x \in I_0 \text{ where this quantity is defined.}$$

When  $M = I_0$ , if  $f$  satisfies the one dimensional definition of non-flatness and  $Sf \leq 0$  (see subsection 3.1 for definitions), then it automatically satisfies these conditions. Now, we are in position to state our main result.

**Theorem A.** *Assume that  $\varphi : \mathbb{T}^1 \times I_0 \rightarrow \mathbb{T}^1 \times I_0$  is a  $C^3$  partially hyperbolic skew product satisfying  $(F_1)$ ,  $(F_2)$  and  $(F_3)$ . If  $\varphi$  is non-uniformly expanding, i.e, for Lebesgue almost every  $z \in \mathbb{T}^1 \times I_0$ ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(z))^{-1}\|^{-1} > 0, \quad (2.1)$$

*then  $\varphi$  admits an absolutely continuous invariant measure. Moreover, if the limit in (2.1) is bounded away from zero, then there exist finitely many ergodic absolutely continuous invariant measures and their basins cover  $M$  up to a zero Lebesgue measure set.*

**Remark 2.1.**

- (i) Note that  $(F_2)$  implies that for any  $z \in M$ ,  $\text{dist}(z, \mathcal{C}) \geq \frac{\text{dist}_{\text{vert}}(z, \mathcal{C})}{2}$ .
- (ii) If  $\mathcal{C}_\theta = \emptyset$  for some  $\theta \in \mathbb{T}^1$  then, as a consequence of  $(F_2)$ ,  $\mathcal{C}_\theta = \emptyset$  for every  $\theta \in \mathbb{T}^1$ . This case is covered by [4, Corollary D], but also follows from (a simple version of) our arguments. For completeness we define  $\text{dist}(z, \emptyset) = 1$ .
- (iii) When the critical set  $\mathcal{C}$  is such that  $\text{dist}(z, \mathcal{C}) \geq \eta \text{dist}_{\text{vert}}(z, \mathcal{C})$  for all  $z \in M$  and some  $\eta > 0$ , then we may replace  $\text{dist}_{\text{vert}}$  by  $\text{dist}$  in the condition  $(F_2)$ .

**2.2. Sequences of smooth one dimensional maps.** In order to prove Theorem A, we analyze the dynamics of the transformation along the family of vertical leaves. The main technical point is to bound the distortion of the iterates along suitable subintervals of the leaves. The precise statement is given in Theorem B. Beforehand, we need to introduce some notations.

Given an interval  $I_0$ , let us consider a sequence  $\{f_k\}_{k \geq 0}$  of  $C^1$  maps  $f_k : I_0 \rightarrow I_0$ . Let us denote by  $\mathcal{C}_k$  the set of critical points of  $f_k$ , for every  $k \geq 0$ . Notice that  $\mathcal{C}_k$  could be an empty set for any  $k \in \mathbb{N}$ . We are interested on the study of the dynamics given by the compositions of maps in the sequence. Thus, we define for  $i \geq 1$  and  $x \in I_0$ ,

$$f^i(x) = f_{i-1} \circ \dots \circ f_1 \circ f_0(x)$$

and we denote  $f^0(x) = x$  for  $x \in I_0$ .

Based on the definitions of  $T_i(x)$  and  $r_i(x)$  on the case that there are just iterates of a function (see for instance [14, page 335]), we define for  $i \in \mathbb{N}$  and  $x \in I_0$ :

$$\begin{aligned} T_i(\{f_k\}, x) &:= \text{Maximal interval contained in } I_0, \text{ containing } x, \\ &\quad \text{such that } f^j(T_i(x)) \cap \mathcal{C}_j = \emptyset \text{ for } 0 \leq j < i; \\ L_i(\{f_k\}, x), R_i(\{f_k\}, x) &:= \text{Connected components of } T_i(\{f_k\}, x) \setminus \{x\}; \\ r_i(\{f_k\}, x) &:= \min \left\{ |f^i(L_i(\{f_k\}, x))|, |f^i(R_i(\{f_k\}, x))| \right\}. \end{aligned}$$

When it does not lead to confusion, we denote these functions just by  $T_i(x)$ ,  $L_i(x)$ ,  $R_i(x)$ ,  $r_i(x)$ . In this subsection and in the proof of the results of this subsection, we will use this simplified notation, since the sequence  $\{f_k\}$  is fixed.

Our goal is to show that positive Lyapunov exponents imply that the average of the  $r_i$  is positive. We consider a sequence  $\{f_k\}$  with positive Lyapunov exponents. Namely,  $\{f_k\}$  satisfies the following condition: there exists  $\lambda > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda \quad (2.2)$$

for every  $x$  in some subset of  $I_0$ .

The following compactness condition on the sequence of maps  $\{f_k\}_{k \geq 0}$ , together with positive Lyapunov exponents, guarantee the positiveness of the average of the  $r_i$ .

Recall that a sequence  $\{f_k\}_k$  of  $C^1$  maps  $f_k : I_0 \rightarrow I_0$  is said to be  $C^1$ -uniformly equicontinuous if, given  $\zeta > 0$ , there exists  $\epsilon > 0$  such that

$$|x - y| < \epsilon \quad \text{implies} \quad \begin{cases} |f_k(x) - f_k(y)| < \zeta \\ |Df_k(x) - Df_k(y)| < \zeta \end{cases} \quad (2.3)$$

for all  $k \in \mathbb{N}$ . Recall also that a sequence  $\{f_k\}_k$  of  $C^1$  maps  $f_k : I_0 \rightarrow I_0$  is said to be  $C^1$ -uniformly bounded if there exists  $\Gamma > 0$  such that for every  $x \in I_0$ ,

$$|f_k(x)|, |Df_k(x)| \leq \Gamma \quad (2.4)$$

for all  $k \in \mathbb{N}$ .

Our main result in this setting is the following.

**Theorem B.** *Let  $\{f_k\}$  be a  $C^1$ -uniformly equicontinuous and  $C^1$ -uniformly bounded sequence of smooth maps  $f_k : I_0 \rightarrow I_0$  for which  $p = \sup_k \# \mathcal{C}_k < \infty$ , and (2.2) holds for all  $x$  in a set  $H$ , for some  $\lambda > 0$ . Then, there exists  $\varsigma > 0$  such that*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(\{f_k\}, x) \geq \varsigma \quad (2.5)$$

for Lebesgue almost every  $x \in H$ .

**Remark 2.2.** We do not require that  $f_k$  be a multimodal map, for any  $k \geq 0$ . The non-positive Schwarzian derivative condition is not necessary.

This result may be viewed as a “random” version of Theorem V.3.2 (page 371) in de Melo, van Strien [14]. Notice however, that this does not follow from the result of de Melo and van Strien because the dynamics of the maps we consider is more complicated. For example, in the unimodal case the hypothesis ensures that the critical point is not periodic, in our context one can not prevent the iterates of the critical set from intersecting the critical set.

Notice that in the setting of Theorem A, the result of Theorem B is applied to the restrictions of  $\varphi$  to the orbits of the vertical leaves.

The result of Theorem B still holds replacing  $\liminf$  by  $\limsup$ .

**Corollary 2.1.** *Let  $\{f_k\}$  be a  $C^1$ -uniformly equicontinuous and  $C^1$ -uniformly bounded sequence of smooth maps  $f_k : I_0 \rightarrow I_0$  for which  $p = \sup_k \# \mathcal{C}_k < \infty$ , and there exists  $\lambda > 0$  such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda \quad (2.6)$$

for all  $x$  in a set  $H$ . Then, there exists  $\varsigma > 0$  such that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(\{f_k\}, x) \geq \varsigma$ , for Lebesgue almost every  $x \in H$ .

In the case that the sequence  $\{f_k\}_{k \geq 0}$  is constant ( $f_k = f$ , for all  $k \geq 0$ ), we obtain the following result for multimodal maps. For definitions involved, see Subsection 3.1.

**Corollary 2.2.** *Let  $f : I_0 \rightarrow I_0$  be a  $C^3$  multimodal map with non-flat critical points. Assume that  $f$  does not have neutral periodic points. If (1.1) holds for Lebesgue almost every point, then there exists an absolutely continuous invariant measure.*

Notice that the hypothesis is weaker than in Keller [11], because we make no assumption on the Schwarzian derivative. On the other hand, we only prove existence (not finiteness) of the absolutely continuous invariant measure.

In particular, for  $C^3$  multimodal maps with non-flat critical points and with eventual negative Schwarzian derivative (i.e, there exists  $k \in \mathbb{N}$  such that  $f^k$  has negative Schwarzian derivative), positive Lyapunov exponents implies the existence of an absolutely continuous invariant measure. Indeed, since for these class of maps, the neutral periodic points are attracting points (see [24, Theorem 2.5]), we can apply Corollary 2.2.

### 3. PRELIMINARY RESULTS

We first recall some well-known properties and tools for one dimensional maps to be used in this work.

**3.1. One dimensional dynamics.** Let  $I$  be an interval and let  $f : I \rightarrow I$  be a differentiable map. A point  $c \in I$  is called a *critical point* if  $f'(c) = 0$ . A map is called *smooth* if it is at least a  $C^1$  map with any number (possibly zero) of critical points. A map is called *multimodal* if it is a smooth map and there is a partition of  $I$  in finitely many subintervals on which  $f$  is strictly monotone. It is called *unimodal* if the partition has exactly two subintervals. Without loss of generality it is assumed that for a multimodal map  $f$ ,  $f(\partial I) \subset \partial I$ . Let  $c_1, \dots, c_d$  be the critical points of  $f$ . We say that the critical point  $c_i$  is  $C^n$  *non-flat of order  $l_i > 1$*  if there exist a local  $C^n$  diffeomorphism  $\phi_i$  with  $\phi_i(c_i) = 0$ , such that near  $c_i$ ,  $f$  can be written as

$$f(x) = \pm |\phi_i(x)|^{l_i} + f(c_i).$$

The critical point is  $C^n$  *non-flat* if it is  $C^n$  non-flat of order  $l_i$  for some  $l_i > 1$ . In all that follows, we will just say that  $c_i$  is a non-flat critical point of a  $C^n$  multimodal map  $f$  if  $c_i$  is a  $C^n$  non-flat critical point. Here  $n = 3$  is enough for Corollary 2.2.

When the map  $f$  is  $C^3$  (or three times differentiable) we can define

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

for  $x$  such that  $f'(x) \neq 0$ . This quantity is called the *Schwarzian derivative* of  $f$  at the point  $x$ . There are many results for one dimensional dynamics that are only known for those maps whose Schwarzian derivative is non-positive.

One standard way to prove the existence of absolutely continuous invariant measures for  $f$  is to define a Markov map associated to  $f$  and take advantage of the known fact of the existence of this kind of measures for Markov maps.

**Definition 3.1.** We call a map  $F : J \rightarrow J$  *Markov* if there exists a countable family of disjoint open intervals  $\{J_i\}_{i \in \mathbb{N}}$  with  $\text{Leb}(J \setminus \cup J_i) = 0$ , such that:

(M<sub>1</sub>) there exists  $K > 0$  such that for every  $n \in \mathbb{N}$  and every  $T$  such that  $F^j(T)$  is contained in some  $J_i$  for  $j = 0, 1, \dots, n$ , it holds

$$\frac{|DF^n(x)|}{|DF^n(y)|} \leq K \quad \text{for } x, y \in T;$$

(M<sub>2</sub>) if  $F(J_k) \cap J_i \neq \emptyset$  then  $J_i \subset F(J_k)$ ;

(M<sub>3</sub>) there exists  $r > 0$  such that  $|F(J_i)| \geq r$  for all  $i$ .



Condition  $(M_1)$  is known as bounded distortion. Given open intervals  $J \subset T$ , let  $L, R$  be the connected components of  $T \setminus J$ . We say that  $T$  is a  $\kappa$ -scaled neighborhood of  $J$  if both connected components of  $T \setminus J$  have length  $\kappa|J|$ . We define  $b(T, J) = |J||T|/|L||R|$ , and when  $f$  is monotone continuous,  $B(f, T, J) = b(f(T), f(J))/b(T, J)$  (this is known as *cross ratio operator*). Koebe Principle claims that the control of cross ratio operator plus  $\kappa$ -scalation (for some  $\kappa > 0$ ) imply bounded distortion (see [14, Theorem IV.1.2]). When  $Sf \leq 0$ , cross ratio satisfies the condition required on Koebe Principle. In order to control the distortion when we consider iterates of a single map without Schwarzian derivative assumptions, we use the next result. Recall that a periodic point  $p$  of period  $k$  is *repelling* if  $|Df^k(p)| > 1$ , *attracting* if  $|Df^k(p)| < 1$  and *neutral* if  $|Df^k(p)| = 1$ . The proof of the result follows from [12, Theorem B] for the unimodal case, and [20, Theorem C] for the multimodal case. The hypothesis of these theorems are less restrictive than ours.

**Theorem 3.1.** *Let  $f : I \rightarrow I$  be a  $C^3$  multimodal map with non-flat critical points. Assume that the periodic points of  $f$  are repelling. Then, there exists  $C > 0$  such that if  $I \subset M$  are intervals and  $f|_M^n$  is a diffeomorphism,*

$$B(f^n, M, I) \geq \exp(-C|f^n(M)|^2).$$

Finally let us state the following theorem which we use in the proof of Corollary 2.2. Recall that an interval  $J \subset I$  is called a *wandering interval* for  $f : I \rightarrow I$  if the intervals  $J, f(J), \dots$  are pairwise disjoint and the images  $f^n(J)$  do not converge to a periodic attractor when  $n \rightarrow \infty$ . De Melo, van Strien and Martens [15] proved that, if  $f : I \rightarrow I$  is a  $C^2$  map with non-flat critical points then  $f$  has no wandering interval. Recall also that the Lebesgue measure is said to be *ergodic* for  $f : J \rightarrow J$ , if for each  $X \subset J$  such that  $f^{-1}(X) = X$ , one of the sets  $X$  or  $\mathbb{C}X$  have full Lebesgue measure.

**Theorem 3.2.** *Let  $f : I \rightarrow I$  be a  $C^3$  map without wandering intervals and with all the periodic points repelling (i.e,  $f$  does not have either attracting or neutral periodic points). Then:*

- (i) *the set of preimages of the critical set  $\mathcal{C}$  is dense in  $I$ .*

Moreover, if the map  $f$  is multimodal then:

- (ii) *every non-wandering critical point is approximated by periodic points;*
- (iii) *if the critical points are non-flat: there are finitely many forward invariant sets  $X_1, \dots, X_k$  such that  $\cup B(X_i)$  has full measure in  $I$ , and  $f|_{B(X_i)}$  is ergodic with respect to the Lebesgue measure (here,  $B(X_i) = \{y; \omega(y) = X_i\}$  is the basin of  $X_i$ ). In the unimodal case we have  $k = 1$ , so  $f$  is ergodic with respect to Lebesgue measure.*

The proof of item (i) follows from standard arguments. For item (ii), see [25]. The proof of item (iii) is contained in the proof of Theorem E of [20].

On our Theorem B we adapt some tools used on one dimensional dynamics: given a smooth map  $f : I_0 \rightarrow I_0$  and  $x \in I_0$ , for every  $n \in \mathbb{N}$ , let  $T_n(x)$  be, the maximal interval containing  $x$  where  $f^n$  is a diffeomorphism. Let  $r_n(x)$  be the length of the smallest component of  $f^n(T_n(x)) \setminus f^n(x)$ . Koebe Principle guarantees distortion bounds in the orbit of a point  $x$ , if the respective  $r_n(x)$  are not too small. Of course, a lower bound on  $r_n(x)$  implies that the images of the monotonicity intervals are not too small. This gives some idea of the importance of the result of Theorem B.

**3.2. Partial hyperbolicity, slow recurrence.** We call a  $C^1$  mapping  $\varphi : M \rightarrow M$  *partially hyperbolic* endomorphism if there are constants  $0 < a < 1$ ,  $C > 0$  and a continuous decomposition of the tangent bundle  $TM = E^c \oplus E^u$  such that:

- (a)  $\|D\varphi^n(z) v\| > C^{-1}a^{-n}$ , for every unit vector  $v \in E^u(z)$ .
- (b)  $\|D\varphi^n(z) u\| < Ca^n\|D\varphi^n(z) v\|$ , for every pair of unit vectors  $u \in E^c(z)$  and  $v \in E^u(z)$ .

for all  $z \in M$  and  $n \geq 0$ . The subbundle  $E^c$  is called *central* and the  $E^u$  is called *unstable*. Observe that we do not ask invariance of the subbundles. For the skew-product maps that we consider, the

central subbundle is given by the vertical direction. The unstable one is given by the horizontal direction. Notice that the partial hyperbolicity property in our skew-product context means that for all  $(\theta, x) \in \mathbb{T}^1 \times I_0$  and  $n \in \mathbb{N}$ ,

$$\frac{\prod_{i=0}^{n-1} |\partial_x f(\varphi^i(\theta, x))|}{|\partial_\theta g^n(\theta)|} \leq Ca^n. \quad (3.1)$$

Let us remark that in the condition  $(F_2)$  of Theorem A we may put  $\text{dist}_{\text{vert}}(z, \mathcal{C})^\gamma$  (with  $\gamma > 1$ ) instead of  $\text{dist}_{\text{vert}}(z, \mathcal{C})$ , if we had a better domination for  $\varphi$ , namely, if for all  $(\theta, x) \in \mathbb{T}^1 \times I_0$ ,

$$\frac{\prod_{i=0}^{n-1} |\partial_x f(\varphi^i(\theta, x))|^\gamma}{|\partial_\theta g^n(\theta, x)|} \leq Ca^n.$$

Finally, recall that the condition of slow recurrence to the critical set  $\mathcal{C}$  (see [4, Equation (6)]) means that given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for Lebesgue almost every  $x \in M$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(\varphi^j(x), \mathcal{C}) \leq \epsilon,$$

where  $\text{dist}_\delta(\varphi^j(x), \mathcal{C}) = \text{dist}(\varphi^j(x), \mathcal{C})$  if  $\text{dist}(\varphi^j(x), \mathcal{C}) < \delta$  and  $\text{dist}_\delta(\varphi^j(x), \mathcal{C}) = 1$  otherwise.

#### 4. COMPOSITIONS OF SMOOTH ONE DIMENSIONAL MAPS

Here we prove Theorem B. In the sequel we introduce some definitions and state results whose proofs are left to the end of the section. Theorem B follows from these results.

**4.1. Proof of Theorem B.** We begin by introducing some sets useful for the proof of the theorem. Recalling the definitions in subsection 2.2, for every  $n \in \mathbb{N}$  and  $\delta > 0$  we denote by,

$$A_n(\{f_k\}, \delta) := \left\{ x \in I_0 ; \frac{1}{n} \sum_{i=1}^n r_i(x) < \delta^2, \ r_n(x) > 0 \right\}, \quad (4.1)$$

and given  $\lambda > 0$ , we define for  $n \in \mathbb{N}$ ,

$$Y_n(\{f_k\}, \lambda) := \left\{ x ; \frac{1}{n} \log |Df^n(x)| > \lambda \right\}. \quad (4.2)$$

When it does not lead to confusion, we denote these sets by  $A_n(\delta)$  and  $Y_n(\lambda)$ . In fact, we will do it in all this section.

It is clear that (2.5) holds (for  $\varsigma = \delta^2$ ) for Lebesgue almost every  $x \in H$ , if  $|\cap_{n \geq N} (\complement A_n(\delta) \cap Y_n(\lambda)) \cap H|$  converges to  $|H|$ , when  $N \rightarrow \infty$  (where  $|B|$  denotes the Lebesgue measure of  $B$  and  $\complement B$  denotes the complement set of  $B$ ). We claim that, in effect, this happens. Indeed, for every  $N \in \mathbb{N}$ , it holds

$$H \cap \left( \bigcap_{n \geq N} Y_n(\lambda) \right) \cap \complement \left( \bigcup_{n \geq N} A_n(\delta) \cap Y_n(\lambda) \right) \subset H \cap \left( \bigcap_{n \geq N} \complement A_n(\delta) \cap Y_n(\lambda) \right).$$

Since (2.2) holds for all  $x \in H$ ,  $|H \cap (\cap_{n \geq N} Y_n(\lambda))|$  converges to the Lebesgue measure of  $H$ . Thus, to prove our claim we just need to prove that  $|\cup_{n \geq N} A_n(\delta) \cap Y_n(\lambda)|$  converges to zero. For this purpose we will state the following result which is the main lemma for proving Theorem B.

**Lemma 4.1.** *Let  $\{f_k\}$  be a  $C^1$ -uniformly equicontinuous and  $C^1$ -uniformly bounded sequence of smooth maps  $f_k : I_0 \rightarrow I_0$  for which  $p = \sup_k \# \mathcal{C}_k < \infty$ . Then, given  $\lambda > 0$ , there exist  $\delta > 0$  such that*

$$|A_n(\{f_k\}, \delta) \cap Y_n(\{f_k\}, \lambda)| \leq |I_0| \exp(-n\lambda/2) \quad (4.3)$$



for  $n$  big enough. Moreover,  $\delta$  depends only on  $\lambda$ , the modulus of continuity (2.3), the uniform bound  $\Gamma$  in (2.4) and the uniform bound  $p$  for the number of critical points.

*Proof that Theorem B follows from Lemma 4.1.* As we have remarked, Lemma 4.1 clearly implies that

$$\bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} C_{A_n(\delta)} \cap Y_n(\lambda)$$

has full Lebesgue measure in  $H$ . Hence, (2.5) holds for  $\varsigma = \delta^2$ , where  $\delta$  is the constant found on Lemma 4.1. This concludes the proof of Theorem B.  $\square$

**4.2. Connected components of the set  $A_n(\delta)$ .** The proof of Lemma 4.1 relies on bounding the number of connected components of the set  $A_n(\delta)$  whose intersection with  $Y_n(\lambda)$  is non-empty. We define a family of sets related to these components. It seems easier to deal and to count the elements of this family than the components of  $A_n(\delta)$ , and it will be enough for our purposes.

For  $\delta > 0$ ,  $a_i \in \{0, 1\}$  for  $i = 1, 2, \dots, n$ ,

$$C_\delta(a_1, a_2, \dots, a_n) := \{x \in I_0 ; r_i(x) \geq \delta \text{ if } a_i = 1, 0 < r_i(x) < \delta \text{ if } a_i = 0\}$$

Note that every connected component of  $C_\delta(a_1, \dots, a_s, a_{s+1})$  is contained in a connected component of  $C_\delta(a_1, \dots, a_s)$ . Moreover, every connected component of  $C_\delta(a_1, \dots, a_s)$  is a union of connected components (with its boundaries) of  $C_\delta(a_1, \dots, a_s, a_{s+1})$ . Also note (recall the definition of  $T_i(x)$  in subsection 2.2) that for every connected component  $I$  of  $C_\delta(a_1, \dots, a_s)$ , we have  $I \subset T_s(x)$  for all  $x \in I$ .

Given  $x \in I_0$  and  $n \in \mathbb{N}$ , if  $f^i(x) \notin \mathcal{C}_i$  for  $0 \leq i < n$ , we can associate to it a sequence  $\{a_i(x)\}_{i=1}^n$ , according to the last definition, in a natural way:

$$a_i(x) = \begin{cases} 0 & \text{if } 0 < r_i(x) < \delta \\ 1 & \text{if } r_i(x) \geq \delta \end{cases}.$$

For this sequence the inequality  $(a_1(x) + \dots + a_n(x))\delta \leq \sum_{i=1}^n r_i(x)$  is satisfied. In particular, for every  $x \in A_n(\delta)$ , the associated sequence  $\{a_i(x)\}_{i=1}^n$  is such that  $a_1(x) + \dots + a_n(x) < \delta n$ . Therefore, if we define

$$C_n(\delta) := \bigcup_{a_1 + \dots + a_n < \delta n} C_\delta(a_1, \dots, a_n),$$

we conclude that  $A_n(\delta) \subset C_n(\delta)$ .

But in fact, we are interested on the connected components of  $A_n(\delta)$  which intersect the set  $Y_n(\lambda)$ . We will say that a connected component  $J$  of  $A_n(\delta)$  is a connected component of  $A'_n(\delta)$  if  $J \cap Y_n(\lambda) \neq \emptyset$ . Analogously we will say that a connected component  $I$  of  $C_\delta(a_1, a_2, \dots, a_n)$  is a connected component of  $C'_\delta(a_1, a_2, \dots, a_n)$  if  $I \cap Y_n(\lambda) \neq \emptyset$ .

We can associate to each connected component of  $A'_n(\delta)$ , a connected component of  $C'_\delta(a_1, a_2, \dots, a_n)$ , where  $a_1 + a_2 + \dots + a_n < \delta n$ : for a connected component  $J$  of  $A'_n(\delta)$ , there exist  $a_1, \dots, a_n$  (such that  $a_1 + a_2 + \dots + a_n < \delta n$ ) and a connected component  $I$  of  $C'_\delta(a_1, a_2, \dots, a_n)$ , for which  $J \cap I \neq \emptyset$ . Indeed, we can consider  $a_i = a_i(x)$  ( $1 \leq i \leq n$ ) for  $x \in J \cap Y_n(\lambda)$ , and  $I$  the connected component of  $C'_\delta(a_1, a_2, \dots, a_n)$  which contains  $x$ . Thus, we associate to  $J$  the component  $I$ .

We would like to bound the number of connected components of  $A'_n(\delta)$  by the number of connected components of  $C'_\delta(a_1, a_2, \dots, a_n)$ , varying  $a_1, \dots, a_n$  such that  $a_1 + a_2 + \dots + a_n < \delta n$ . But every connected component of  $C'_\delta(a_1, a_2, \dots, a_n)$  (with  $a_1 + a_2 + \dots + a_n < \delta n$ ) could intersect more than one connected component of  $A'_n(\delta)$ . By this reason we define the following set:

$$A''_n(\delta) := \bigcup_{J' \in A'_n(\delta)} J'',$$

where

$$J'' := J' \cup \bigcup_{a_1 + \dots + a_n < \delta n} \{\text{connected components of } C_\delta(a_1, \dots, a_n) \text{ which intersect } J' \cap Y_n(\lambda)\}$$

Obviously, a connected component of  $A_n''(\delta)$  could contain more than one connected component of  $A_n'(\delta)$ . However, the restriction of  $f^n$  to every connected component of  $A_n''(\delta)$  is a diffeomorphism. Using this fact, we will show in the proof of Lemma 4.1 that in order to obtain (4.3), it is enough to estimate the number of connected components of  $A_n''(\delta)$ .

Since every component of  $A_n''(\delta)$  intersect at least one component of  $C'_\delta(a_1, \dots, a_n)$ , we conclude that

$$\# A_n''(\delta) \leq \sum \# C'_\delta(a_1, \dots, a_n) \quad (4.4)$$

where the sum is over all  $a_1, \dots, a_n$  such that  $a_1 + \dots + a_n < \delta n$ , and  $\#X$  denotes the number of connected components of  $X$ .

As we have said, Lemma 4.1 is a consequence of the following result, which gives an estimate of the number of connected components of  $A_n''(\delta)$ .

**Lemma 4.2.** *Given  $\lambda > 0$ , there exists  $\delta > 0$  such that the number of connected components of  $A_n''(\delta)$  is less than  $\exp(n\lambda/2)$ . Moreover,  $\delta$  depends only on  $\lambda$ , the modulus of continuity (2.3), the uniform bound  $\Gamma$  in (2.4) and the uniform bound  $p$  for the number of critical points.*

**4.3. Consequences of expansion and continuity.** For the proof of Lemma 4.2 we will use several results that we state now. First we give some notations. Given  $\epsilon > 0$ , for every  $k \geq 0$ , we call  $V_\epsilon \mathcal{C}_k$  a neighborhood of  $\mathcal{C}_k$  defined as the union of all  $B(x, \epsilon)$  (ball centered in  $x$  of ratio  $\epsilon$ ) varying  $x \in \mathcal{C}_k$ . In order to simplify the notation we say that  $f^j(x) \in V_\epsilon \mathcal{C}$  if  $f^j(x) \in V_\epsilon \mathcal{C}_j$  for any  $j \in \mathbb{N}$ . The next lemma asserts that for points in  $Y_n(\lambda)$ , the frequency of visits to the neighborhood  $V_\epsilon \mathcal{C}$  can be made arbitrarily small, if  $\epsilon$  is chosen small enough.

**Lemma 4.3.** *Given  $\gamma > 0$ , there exists  $\epsilon > 0$ , such that for  $x \in Y_n(\lambda)$ ,*

$$\frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \mathcal{C}}(f^j(x)) < \gamma.$$

Moreover,  $\epsilon$  does not depend on  $n$ , but it depends on  $\lambda$ , on the modulus of continuity of  $\{f_k\}$  and on the uniform bound of  $\{Df_k\}$ .

*Proof.* Using the fact that the sequence  $\{f_k\}_{k \geq 0}$  is  $C^1$ -uniformly equicontinuous, we conclude that given  $\zeta > 0$ , there exists  $\epsilon = \epsilon(\zeta)$  such that

$$|x - \mathcal{C}_k| < \epsilon \quad \text{implies} \quad |Df_k(x)| < \zeta \quad \text{for all } k \geq 0. \quad (4.5)$$

On the other hand, since  $\{f_k\}_{k \geq 0}$  is  $C^1$ -uniformly bounded,  $|Df_k(x)| \leq \Gamma$  for all  $k \geq 0$  and  $x \in I_0$ . Thus,  $\log |Df_j(f^j(x))| < \log \zeta$  if  $f^j(x) \in V_\epsilon \mathcal{C}$  and  $\log |Df_j(f^j(x))| \leq \log \Gamma$  otherwise.

Since  $\lambda n < \sum_{j=0}^{n-1} \log |Df_j(f^j(x))|$  for  $x \in Y_n(\lambda)$  and  $\log \zeta \rightarrow -\infty$  when  $\zeta \rightarrow 0$ , there must exist  $\epsilon$  as stated.  $\square$

**Corollary 4.1.** *Assume that for Lebesgue almost every  $x \in I_0$*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| \geq \lambda > 0.$$

*Then, given  $\gamma > 0$ , there exists  $\epsilon > 0$ , such that for Lebesgue almost every  $x \in M$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \mathcal{C}}(f^j(x)) < \gamma.$$

Let us denote for  $i, j \in \mathbb{N}$ , and  $x \in I_0$ ,

$$f_i^j(x) = f_{i+j-1} \circ \dots \circ f_{i+1} \circ f_i(x)$$

and  $f_0^0(x) = x$ . Notice that  $f_0^j(x) = f^j(x)$  for  $j \geq 0$  and  $x \in I_0$ . Again by the  $C^1$ -uniform equicontinuity of the sequence  $\{f_k\}$ , we have the following property.

**Lemma 4.4.** *Given  $\epsilon > 0$  and  $l \in \mathbb{N}$ , there exists  $\delta = \delta(l)$  such that*

$$|x - y| \leq 2\delta \quad \text{implies} \quad |f_i^j(x) - f_i^j(y)| < \epsilon \quad (4.6)$$

for all  $i \geq 0$  and  $0 \leq j \leq l$ . Moreover,  $\delta$  just depends (on  $l, \epsilon$  and) on the modulus of continuity of  $\{f_k\}$ .

**Remark 4.1.** When  $l \rightarrow \infty$  then  $\delta(l) \rightarrow 0$ . Observe that we also have: given  $\epsilon > 0$  and  $\delta > 0$ , there exists  $l = l(\delta) \in \mathbb{N}$  such that (4.6) holds for  $0 \leq j \leq l$ .

From now on,  $\#\{I \subset C_\delta(a_1, \dots, a_n); I \text{ satisfies the property P}\}$  denotes the number of connected components of  $C_\delta(a_1, \dots, a_n)$  which satisfy the property P.

In order to count the components whose intersection with  $Y_n(\lambda)$  is non-empty, let us decompose this set in a convenient way. Given  $\epsilon > 0$ ,  $m \leq n$ ,  $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$ , we define

$$Y_{n,\epsilon}(t_1, \dots, t_m) = \{x \in Y_n(\lambda); f^j(x) \in V_\epsilon \mathcal{C} \text{ if and only if } j \in \{t_1, \dots, t_m\}\}$$

By Lemma 4.3 we conclude that given  $\gamma > 0$ , there exists  $\epsilon > 0$  such that

$$Y_n(\lambda) = \bigcup_{m=0}^{\gamma n} \bigcup_{t_1, \dots, t_m} Y_{n,\epsilon}(t_1, \dots, t_m) \quad (4.7)$$

where the second union is over all subsets  $\{t_1, \dots, t_m\}$  of  $\{0, 1, \dots, n-1\}$ . This together with (4.4) yields,

$$\#A_n''(\delta) \leq \sum_{a_1, \dots, a_n} \sum_{t_1, \dots, t_m} \#\{I \subset C_\delta(a_1, \dots, a_n); I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} \quad (4.8)$$

where the first sum is over all  $a_1, \dots, a_n$  such that  $a_1 + \dots + a_n < \delta n$  and the second one is over all subsets  $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$  with  $m < \gamma n$ .

**4.4. Connected components of  $C_\delta(a_1, \dots, a_s)$ .** To prove Lemma 4.2 we just need to bound the double sum in (4.8). For this we will show some claims related to the number of connected components of the sets  $C_\delta(a_1, \dots, a_n)$ . Recall that  $p$  is the maximum number of elements in any  $\mathcal{C}_k$  (for  $k \geq 0$ ). Given  $I \subset I_0$  and  $s \in \mathbb{N}$ , we say  $f^s(I) \cap \mathcal{C} = \emptyset$  (resp.  $\neq \emptyset$ ) if  $f^s(I) \cap \mathcal{C}_s = \emptyset$  (resp.  $\neq \emptyset$ ).

**Claim 4.1.** *For any  $a_1, a_2, \dots, a_s$  with  $a_j \in \{0, 1\}$  for all  $j$ ,*

$$\#C_\delta(a_1, \dots, a_s, 0) + \#C_\delta(a_1, \dots, a_s, 1) \leq 3(p+1)\#C_\delta(a_1, \dots, a_s)$$

**Claim 4.2.** *Let  $s, n \in \mathbb{N}$  and  $J$  be a component of  $C_\delta(a_1, \dots, a_s, 0)$ . If  $f^{s+i}(J) \cap \mathcal{C} = \emptyset$  for  $1 \leq i \leq n$ , then*

$$\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0^{i+1}), I \subseteq J\} \leq i+1.$$

for  $1 \leq i \leq n$ , where  $0^{i+1}$  means that the last  $i+1$  terms are equal to 0.

To bound the number of connected components whose intersection with  $Y_{n,\epsilon}(t_1, \dots, t_m)$  is non-empty, we have the following claim.

**Claim 4.3.** *Let  $l \in \mathbb{N}$  and  $\epsilon > 0$  be constants and let  $\delta = \delta(l)$  be the number given by Lemma 4.4. For any  $a_1, \dots, a_s$  with  $a_j \in \{0, 1\}$ ,  $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$ . If  $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$  and  $i \leq l$ , then*

$$\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0^{i+1}), I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} \leq (i+1)\#\{I \subseteq C_\delta(a_1, \dots, a_s, 0), I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\}.$$

*Proof of Lemma 4.2.* We prove the lemma assuming the claims above. We have basically four constants, namely,  $\delta, \gamma, \epsilon, l$ . It is very important the order in what they are chosen. First, we choose  $l \in \mathbb{N}$  according to the equation (4.12), then we choose  $\gamma > 0$  according to (4.13). Next, we find  $\epsilon > 0$ , using Lemma 4.3, in such a way that (4.7) holds. Finally, given  $\epsilon$  and  $l$ , let  $\delta > 0$  be the constant given by Lemma 4.4 and satisfying (4.14).

Given  $m < n, \delta > 0$  and  $\epsilon > 0$ , let us consider  $a_1, \dots, a_n$  with  $a_i \in \{0, 1\}$  (such that  $a_1 + a_2 + \dots + a_n < \delta n$ ) and  $\{t_1, \dots, t_m\} \subset \{0, \dots, n-1\}$ . We can decompose the sequence  $a_1 \dots a_n$  in maximal blocks of 0's and 1's. We write the symbol  $\xi$  in the  $j$ -th position if  $a_j = 1$  or,  $a_j = 0$  and  $j = t_k$  for some  $k \in \{1, \dots, m\}$ . In this way we have,

$$a_1 a_2 \dots a_n = \xi^{i_1} 0^{j_1} \xi^{i_2} 0^{j_2} \dots \xi^{i_h} 0^{j_h} \quad (4.9)$$

with  $0 \leq i_k, j_k \leq n$  for  $k = 1, \dots, h$ ,  $\sum_{k=1}^h (i_k + j_k) = n$  and  $\sum_{k=1}^h i_k < m + \delta n$ .

Lets us assume that  $a_1, \dots, a_n$  are as in (4.9). Let  $l, \epsilon$  and  $\delta$  be as in Lemma 4.4. Using claims 4.1 and 4.3 we have,

$$\begin{aligned} \#\{I \subset C_\delta(a_0, \dots, a_n), I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} &\leq \\ &\leq (3(p+1)(l+1)^{\frac{1}{l}+1}(3(p+1))^{i_1}) \dots (3(p+1)(l+1)^{\frac{1}{l}+1}(3(p+1))^{i_h}) \\ &\leq (3(p+1))^{\sum_{k=1}^h i_k} (3(p+1))^h (l+1)^{\frac{\sum_{k=1}^h i_k}{l} + h} \\ &\leq (3(p+1))^{m+\delta n+h} (l+1)^{\frac{m}{l}+h}. \end{aligned}$$

Let us remark some useful properties about the decomposition (4.9):

- if  $m < \gamma n$  then, since  $a_1 + a_2 + \dots + a_n < \delta n$ , we have that  $\sum_{k=1}^h i_k < \gamma n + \delta n$ ;
- if  $a_1 + a_2 + \dots + a_n < \delta n$  and  $m < \gamma n$ , the number of blocks  $\xi^{i_k} 0^{j_k}$  is bounded by the sum of these quantities, i.e,  $h < (\delta + \gamma)n + 1$ .

Therefore, if  $a_1 + a_2 + \dots + a_n < \delta n$  and  $m < \gamma n$  we conclude from the inequality above that for  $n$  big enough,

$$\begin{aligned} \#\{I \subset C_\delta(a_1, \dots, a_n), I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} \\ \leq (3(p+1))^{\gamma n + \delta n} (3(p+1))^{2(\delta+\gamma)n} (l+1)^{\frac{m}{l} + 2(\delta+\gamma)n} \leq \exp(n \psi_0(l, \gamma, \delta)) \end{aligned} \quad (4.10)$$

where  $\psi_0(l, \gamma, \delta) = 3(\delta + \gamma) \log(3(p+1)) + 2(\delta + \gamma + \frac{1}{l}) \log(2l)$ .

On the other hand, by the Stirling's formula, the number of subsets of  $\{0, 1, \dots, n-1\}$  of size less than  $\gamma n$  is bounded by  $\exp(n(\psi_1(\gamma)))$  and  $\psi_1(\gamma) \rightarrow 0$  when  $\gamma \rightarrow 0$ . Therefore, from this fact and (4.10), we conclude

$$\sum_{t_1, \dots, t_m} \#\{I \subset C_\delta(a_1, \dots, a_n); I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset\} \leq \exp(n \psi_2(l, \gamma, \delta)) \quad (4.11)$$

where the sum is over all subset  $\{t_1, \dots, t_m\} \subset \{0, 1, \dots, n-1\}$  with  $m < \gamma n$ , and  $\psi_2(l, \gamma, \delta) = \psi_0(l, \gamma, \delta) + \psi_1(\gamma)$ .

Once again, using the Stirling's formula we conclude that the number of sequences  $a_1, a_2, \dots, a_n$  of 0's and 1's such that  $a_1 + a_2 + \dots + a_n < \delta n$  is less or equal than  $\exp(n\psi_3(\delta))$  with  $\psi_3(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$ . Hence, by (4.8) and (4.11), we have that whenever  $\gamma$  and  $\epsilon$  satisfy (4.7),

$$\#A_n''(\delta) \leq \exp(n \psi_4(l, \gamma, \delta))$$

where

$$\psi_4(l, \gamma, \delta) = 3(\delta + \gamma) \log(3(p+1)) + 2\left(\delta + \gamma + \frac{1}{l}\right) \log(2l) + \psi_1(\gamma) + \psi_3(\delta).$$

Hence, we have to choose  $l$  such that

$$\frac{2}{l} \log(2l) < \frac{\lambda}{14} \quad (4.12)$$

and, let  $\gamma > 0$  be such that

$$\left. \begin{aligned} 2\gamma \log(2l) &< \frac{\lambda}{14} \\ 3\gamma \log(3(p+1)) &< \frac{\lambda}{14} \\ \psi_1(\gamma) &< \frac{\lambda}{14} \end{aligned} \right\}. \quad (4.13)$$

Next, we find  $\epsilon > 0$ , using Lemma 4.3. Finally, given  $\epsilon$  and  $l$ , let  $\delta > 0$  be the constant given by Lemma 4.4 and satisfying

$$\left. \begin{aligned} 2\delta \log(2l) &< \frac{\lambda}{14} \\ 3\delta \log(3(p+1)) &< \frac{\lambda}{14} \\ \psi_3(\delta) &< \frac{\lambda}{14} \end{aligned} \right\}. \quad (4.14)$$

With this choice,  $\psi_4(l, \gamma, \delta) \leq \frac{\lambda}{2}$ . Hence the first part of Lemma 4.2 is proved, assuming the three claims. Now we will prove the claims.  $\square$

#### 4.5. Proof of claims, Lemmas 4.1 and 4.2.

*Proof of Claim 4.1.* Let  $I$  be a connected component of  $C_\delta(a_1, \dots, a_s)$ .

*Case 1.*  $f^s(I) \cap \mathcal{C} = \emptyset$ . In this case,  $I$  is divided at most in 3 connected components of  $C_\delta(a_1, \dots, a_s, 0) \cup C_\delta(a_1, \dots, a_s, 1)$ . Indeed, since  $I \subset T_{s+1}(x)$  for every  $x \in I$ , if  $I' \subset I$  is a component of  $C_\delta(a_1, \dots, a_s, 0)$ , it can not exist one component of  $C_\delta(a_1, \dots, a_s, 1)$  at each side of  $I'$ . Hence, the following situations can occur:

- i) There are two components of  $C_\delta(a_1, \dots, a_s, 0)$  in  $I$ , each of them has one extreme of  $I$ , and between them there is one component of  $C_\delta(a_1, \dots, a_s, 1)$ .
- ii) There is exactly one component of  $C_\delta(a_1, \dots, a_s, 0)$  in  $I$ . In this case there is at most one component of  $C_\delta(a_1, \dots, a_s, 1)$  in  $I$ .
- iii) There are no components of  $C_\delta(a_1, \dots, a_s, 0)$  in  $I$ . In this case  $I$  is a component of  $C_\delta(a_1, \dots, a_s, 1)$ .

*Case 2.*  $f^s(I) \cap \mathcal{C} \neq \emptyset$ . First  $I$  is divided at most in  $p+1$  components, each one with at least one boundary which goes by  $f^s$  to  $\mathcal{C}$ . After that, following the same arguments used in case 1, we conclude that each one of these components is divided at most in 3 components.

*Proof of Claim 4.2.* The proof will be by induction on  $i$ . For  $i = 1$ , it follows by the proof of Claim 4.1. Let us assume that the statement is true for  $j \leq i-1$ . Let  $I_1, \dots, I_t$  be the components of  $C_\delta(a_1, \dots, a_s, 0^{(i-1)+1})$  contained in  $I$ . By the induction hypothesis  $t \leq i$  and we assume that  $f^i(I) \cap \mathcal{C} = \emptyset$ . We claim that there exist at most one  $k \in \{1, \dots, t\}$  such that  $I_k$  is divided in two components of  $C_\delta(a_1, \dots, a_s, 0^{i+1})$  (the others  $I_k$ 's generate one or none component of  $C_\delta(a_1, \dots, a_s, 0^{i+1})$ ). Indeed, if  $I_{k_1}$  and  $I_{k_2}$  are divided in two components of  $C_\delta(a_1, \dots, a_s, 0^{i+1})$ , let  $I_{k_1}^+$  and  $I_{k_1}^-$  be the components of  $C_\delta(a_1, \dots, a_s, 0^{i+1})$  and let  $J_{k_1}$  be the component of  $C_\delta(a_1, \dots, a_s, 0^i, 1)$  contained on  $I_{k_1}$ . Analogously, let  $I_{k_2}^+$ ,  $I_{k_2}^-$ ,  $J_{k_2}$  be the corresponding components for  $I_{k_2}$ . Two of the  $I_{k_j}^*$  ( $j \in \{1, 2\}, * \in \{+, -\}$ ) are between  $J_{k_1}$  and  $J_{k_2}$ . This is a contradiction because  $r_{s+i+1}(x) < \delta$  for  $x \in I_{k_j}^*$  and  $r_{s+i+1}(x) \geq \delta$  for  $x \in J_{k_1} \cup J_{k_2}$ . Hence, there are at most  $i+1$  components of  $C_\delta(a_1, \dots, a_s, 0^{i+1})$  contained in  $I$ .  $\square$

*Proof of Claim 4.3.* Let  $I$  be a connected component of  $C_\delta(a_1, \dots, a_s, 0)$ . Then we have  $|f^{s+1}(I)| \leq 2\delta$ , and by Lemma 4.4,  $|f^{s+i}(I)| < \epsilon$  for  $i \leq l+1$ . If  $f^{s+j}(I) \cap \mathcal{C} \neq \emptyset$  for some  $j \leq i$ , then for all  $x \in I$ ,  $f^{s+j}(x) \in V_\epsilon \mathcal{C}$ . Since  $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_m\} = \emptyset$ , then  $I \cap Y_{n,\epsilon}(t_1, \dots, t_m) = \emptyset$ .

Hence, if  $I \cap Y_{n,\epsilon}(t_1, \dots, t_m) \neq \emptyset$  and  $\{s+1, \dots, s+i\} \cap \{t_1, \dots, t_l\} = \emptyset$ , then  $f^{s+j}(I) \cap \mathcal{C} = \emptyset$  for all  $1 \leq j \leq i$ . The result follows using Claim 4.2.  $\square$

*End of proof of Lemma 4.2.* We have proved the existence of  $\delta$  (given  $\lambda$ ) such that the number of connected components of  $A_n''(\delta)$  is less than  $\exp(n\lambda/2)$ . On the other hand, observe that the choice of  $\delta$  is given fundamentally by Lemmas 4.3 and 4.4. Namely,  $\delta$  depends on: the constant  $\lambda$  in the definition of  $Y_n(\lambda)$ ; the uniformity of  $\epsilon$  (given  $\zeta > 0$ ) on the equation (4.5); the uniform boundedness of  $|Df_k|$  on the proof of Lemma 4.3; the uniformity of  $\delta$  (given  $\epsilon$  and  $l$ ) on the equation (4.6); and the uniform boundedness of the number of critical points for  $f_k$ , where  $k \geq 0$ . So,  $\delta$  depends only on the modulus of continuity (2.3), the uniform bound  $\Gamma$  in (2.4) and the uniform bound  $p$  for the cardinal of the set of critical points, as stated. This concludes the proof of Lemma 4.2.  $\square$

Finally we will prove that Lemma 4.1 follows as a consequence of Lemma 4.2.

*Proof of Lemma 4.1.* Note that if  $J''$  is a connected component of  $A_n''(\delta)$  then  $f^n$  restricted to  $J''$  is a diffeomorphism onto its image. Since the set  $Y_n(\lambda)$  is an open subset of  $I_0$ , there exist at most countably many components  $\{I_k\}_{k \in \mathbb{N}}$  of  $Y_n(\lambda) \cap A_n(\delta)$  on  $J''$ . For all  $k \in \mathbb{N}$ ,

$$|I_k| < (\exp(-n\lambda))|f^n(I_k)|,$$

since for every  $x \in I_k$ ,  $|Df^n(x)| > \exp(\lambda n)$ . Adding these inequalities ( $k \in \mathbb{N}$ ),

$$|\cup_k I_k| < (\exp(-n\lambda)) \sum_k |f^n(I_k)| \leq (\exp(-n\lambda))|f^n(J'')|.$$

Then, since  $|f^n(J'')|$  is bounded by  $|I_0|$ ,

$$|(A_n(\delta) \cap J'') \cap Y_n(\lambda)| < |I_0| \exp(-n\lambda)$$

for every connected component  $J''$  of  $A_n''(\delta)$ . To finish the proof of this lemma it is enough to use the estimate of the number of components of  $A_n''(\delta)$  given by Lemma 4.2. The statement about the dependence of  $\delta$  follows from the analogous conclusion on Lemma 4.2.  $\square$

## 5. CONSEQUENCES OF THEOREM B

We prove Corollaries 2.1 and 2.2. Recall that this last result deals with only one single interval map.

**5.1. Proof of Corollary 2.1.** Since (2.6) holds for all  $x \in H$ ,  $H \subset \cup_{k \geq n} Y_k(\lambda)$  (for any  $n \in \mathbb{N}$ ). Thus

$$\left| \left( \bigcap_{k \geq n} A_k(\delta) \cup \bigcap_{k \geq n} Y_k(\lambda) \right) \cap H \right| \leq \left| \left( \bigcap_{k \geq n} A_k(\delta) \cup \bigcap_{k \geq n} Y_k(\lambda) \right) \cap \bigcup_{k \geq n} Y_k(\lambda) \right| \leq \sum_{k=n}^{\infty} |A_k(\delta) \cap Y_k(\lambda)|$$

for any  $n \in \mathbb{N}$ . By Lemma 4.1, for any  $\epsilon > 0$ , the last sum is less than  $\epsilon$  if  $n \geq N(\epsilon)$ . This implies that  $|(\cap_{n \geq N(\epsilon)} \cup_{k=n}^{\infty} A_k(\delta) \cap Y_k(\delta)) \cap H| \geq |H| - \epsilon$ . This means that the set

$$\left\{ x \in H; \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(x) \geq \delta^2 \right\}$$

has Lebesgue measure greater than  $|H| - \epsilon$ . Since this can be done for any  $\epsilon > 0$ , the corollary follows with  $\zeta = \delta^2$ .  $\square$



**5.2. Proof of Corollary 2.2.** The proof that we give is similar to the proof by de Melo and van Strien [14, Theorem V.3.2] for Keller's theorem. We also construct a Markov map  $F$  induced by  $f$ .

*Proof of Corollary 2.2.* By Theorem 3.2 (item (iii)) and Corollary 2.1 (applied to  $f_n = f$  for  $n \geq 0$ ),

$$X = \left\{ x \in I_0; \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n r_i(x) \geq \varsigma \right\}$$

has full Lebesgue measure for some  $\varsigma > 0$ .

Let us consider a partition  $\mathcal{P}$  of  $I_0$  into (a finite number of) subintervals, with norm less than  $\varsigma/4$  and such that the set of extremes of such subintervals is forward invariant. The existence of this partition follows from Theorem 3.2 (items (i) and (ii)). Let  $\varsigma'$  be the minimum of the lengths of the elements of  $\mathcal{P}$ . For every  $x \in I_0$ , we denote by  $J(x)$  the subinterval of the partition which contains  $x$ . And for every  $J \in \mathcal{P}$ , let us denote by  $J^-$  (resp.  $J^+$ ) the rightmost (resp. leftmost) subinterval of the partition next to  $J$ . We choose  $N \in \mathbb{N}$  such that the intervals of monotonicity of  $f^n$  have length less than  $\varsigma'/4$ , for  $n \geq N$ .

Given  $x \in X$ , there are infinitely many  $k$ 's such that  $r_k(x) > \varsigma/2$ . Let  $k(x) \geq N$  be minimal such that

$$f^{k(x)}(T_{k(x)}(x)) \supset J(f^{k(x)}(x)) \cup J(f^{k(x)}(x))^+ \cup J(f^{k(x)}(x))^- , \quad (5.1)$$

and consider  $I(x) \subset T_{k(x)}(x)$  such that  $f^{k(x)}(I(x)) = J(f^{k(x)}(x))$ . Obviously, for every  $y \in I(x)$ ,  $k(y) \leq k(x)$ ; and using the forward invariance of the set of extremes of the subintervals of  $\mathcal{P}$ , we conclude that in fact,  $k(y) = k(x)$  and  $I(y) = I(x)$ . Hence, we can define  $F : \cup_{x \in X} I(x) \rightarrow \cup_{J \in \mathcal{P}} J$ , by  $F|_{I(x)} = f^{k(x)}|_{I(x)}$ . We claim that this map is Markov (recall Definition 3.1). Indeed,  $(M_3)$  is satisfied because  $|F(I(x))| = |J(f^{k(x)}(x))| \geq \varsigma'$ . Since  $I(x)$  does not contain extremes of subintervals of  $\mathcal{P}$  in its interior,  $I(x)$  is completely contained on some element of  $\mathcal{P}$ . This implies that  $(M_2)$  holds.

By Theorem 3.1,  $B(f^{k(x)}, T, M) \geq K'$  for any  $M \subset T \subset T_{k(x)}$ . On the other hand, by (5.1),  $f^{k(x)}(T_{k(x)}(x))$  contains a neighborhood  $\tau$ -scaled of  $f^{k(x)}(I(x))$ , where  $\tau = 4\varsigma'/\varsigma$ . Hence, by Koebe Principle (see [14, Theorem IV.1.2]),  $F$  has bounded distortion on  $I(x)$ . It remains to show bounded distortion for the iterates of  $F$ . Given  $x \in X$  and  $s \in \mathbb{N}$ ; let  $m(s, x) \in \mathbb{N}$  be such that  $F^s(x) = f^{m(s, x)}(x)$  and let  $I_s(x)$  be the domain of  $F^s$  containing  $x$ . By the choice of  $N$ , since  $m(s, x) \geq N$ ,  $T_{m(s, x)}(x)$  is contained in at most two elements of  $\mathcal{P}$ . Using this and (5.1) we can prove inductively that for  $x \in X$  and  $s \geq 1$ ,

$$f^{m(s, x)}(T_{m(s, x)}(x)) \supset J(f^{m(s, x)}(x)) \cup J(f^{m(s, x)}(x))^+ \cup J(f^{m(s, x)}(x))^- .$$

So,  $(M_1)$  holds and  $F$  is a Markov map as we claimed. Hence, there exists an ergodic absolutely continuous invariant measure  $\nu$  for  $F$  (see [14, Theorem V.2.2]). This measure induces an absolutely continuous invariant measure for  $f$  if  $\sum_{i=1}^{\infty} k(i)\nu(I_i) < \infty$  (see [14, Lemma V.3.1]). Assume by contradiction that  $\sum_{i=1}^{\infty} k(i)\nu(I_i) = \infty$ . By Birkhoff's Ergodic Theorem,

$$\frac{n_s(x)}{s} = \frac{k(x) + k(F(x)) + \dots + k(F^s(x))}{s} \rightarrow \int k(x) d\nu(x) = \sum_{i=1}^{\infty} k(i)\nu(I_i) = \infty$$

for  $\nu$ -almost every point  $x$ . For every  $x \in X$  and  $i \in \mathbb{N}$ , if  $n_i(x) \leq n < n_{i+1}(x)$  and  $r_n(x) > \varsigma/2$ , then  $n - n_i(x) < N$ , since in this case  $f^n(T_n(x))$  covers one element of the partition and its two neighbors. Thus we have for  $n_s(x) \leq n < n_{s+1}(x)$ ,

$$\frac{1}{n} \sum_{i=1}^n r_i(x) = \frac{1}{n} \sum_{i, r_i(x) > \varsigma/2} r_i(x) + \frac{1}{n} \sum_{i, r_i(x) \leq \varsigma/2} r_i(x) < \frac{N(s+2)}{n_s(x)} |I_0| + \varsigma/2$$

which implies that  $\limsup_{n \rightarrow \infty} 1/n \sum_{i=1}^n r_i(x) < \varsigma$ . Since it holds for  $\nu$ -almost every  $x$ , it contradicts that  $X$  has full Lebesgue measure. Hence there exists absolutely continuous invariant measure for  $f$ .  $\square$

## 6. HYPERBOLIC-LIKE TIMES

In this section we develop some preparatory tools for the proof of Theorem A. The arguments are independent from the previous sections. We prove a similar behavior of points with  $r_k \geq \sigma$  (for some  $\sigma > 0$ ) and points with  $k$  being one of its  $(\sigma', \delta)$ -hyperbolic times. See Lemma 5.2 of [4] and Proposition 6.3 below. Because of this, if  $r_k(z) \geq \sigma$ , we say  $k$  is a  $\sigma$ -hyperbolic-like time for  $z \in M$ . We need to adapt some notations from subsection 2.2 to the setting defined by Theorem A.

For every  $z = (\theta, x) \in \mathbb{T}^1 \times I_0$ , let us denote by  $T_i(\theta, x)$  (or  $T_i(z)$ ) the function  $T_i(\{f_n\}, x)$  defined on subsection 2.2, considering the sequence  $\{f_n\}_{n \geq 0}$  given by  $f_n = f_{g^n(\theta)}$  for all  $n \geq 0$ . We proceed analogously for  $L_i(\theta, x)$  (or  $L_i(z)$ ),  $R_i(\theta, x)$  (or  $R_i(z)$ ) and  $r_i(\theta, x)$  (or  $r_i(z)$ ). We also define

$$\begin{aligned}\mathcal{T}_i(z) &:= \{\theta\} \times T_i(z); \\ \mathcal{L}_i(z), \mathcal{R}_i(z) &:= \{\theta\} \times L_i(z), \{\theta\} \times R_i(z);\end{aligned}$$

for every  $z = (\theta, x) \in \mathbb{T}^1 \times I_0$  and every  $i \in \mathbb{N}$ . In all the results below we assume that we are in the conditions of Theorem A.

**6.1. Horizontal behavior of dominated skew-products.** One important property of our mappings due to the domination condition is the preservation of the nearly horizontal curves. This means that the iterates of nearly horizontal curves are still nearly horizontal. We state it in a precise way.

**Definition 6.1.** We call  $\widehat{X} \subset \mathbb{T}^1 \times I_0$  a  $t$ -curve if there exists  $J \subset \mathbb{T}^1$  and  $X : J \rightarrow I_0$  such that:  $\widehat{X} = \text{graph}(X)$ ,  $X$  is  $C^1$  and  $|X'(\theta)| \leq t$  for every  $\theta \in J$ .

There exists an analogous definition given by Viana (see [22], section 2.1), but he also asks the second derivative to be less than  $t$ . He calls the curves with these properties *admissible curves*. In his setting he proves that the admissible curves are preserved under iteration.

**Proposition 6.1.** *There exist  $\alpha > 0$  and  $n_0 \in \mathbb{N}$  such that, if  $\widehat{X}$  is an  $\alpha$ -curve and  $\varphi^n(\widehat{X})$  is the graph of a  $C^1$  map, then  $\varphi^n(\widehat{X})$  is an  $\alpha$ -curve, provided that  $n \geq n_0$ . Moreover, there exists  $C_1 = C_1(\alpha)$  such that if  $\widehat{X}$  is a  $\alpha$ -curve, then  $\varphi^n(\widehat{X})$  is a  $C_1$ -curve, for all  $n$ , provided that  $\varphi^n(\widehat{X})$  is a graph.*

*Proof.* Let  $\widehat{X} = \{(\theta, X(\theta)); \theta \in J\}$  be a  $C^1$  curve with  $|X'(\theta)| \leq \alpha$  for every  $\theta \in J$ . Let us define inductively for  $n \geq 1$ ,  $X_n(g^n(\theta)) = f(g^{n-1}(\theta), X_{n-1}(g^{n-1}(\theta)))$ , where  $X_0 = X$ . Thus we can prove by induction that  $\varphi^n(\theta, X(\theta)) = (g^n(\theta), X_n(g^n(\theta)))$ , for  $n \geq 1$ .

Proceeding similarly as in [22, Lemma 2.1], using the partial hyperbolicity (see inequality (3.1)) and considering  $L = \sup(\partial_\theta f / \partial_\theta g)$ , we have that

$$|X'_n(g^n(\theta))| \leq L + \sum_{k=1}^{n-1} LC(a)^k + Ca^n \alpha \leq LCA + Ca^n \alpha$$

for  $n \geq 1$ , where  $A = \sum_{k=0}^{\infty} a^k$ . Hence, for some  $\alpha$  and  $n_0$  big enough,  $|X'_n(g^n(\theta))| \leq \alpha$  for all  $n \geq n_0$ .  $\square$

Since all the iterates of  $\alpha$ -curves are almost horizontal then their lengths are given basically by the derivative of  $\varphi$  in the horizontal direction. We state this in the following result.

**Proposition 6.2.** *Let  $C_1 = C_1(\alpha)$  be the constant found on Proposition 6.1. There exists  $C_2 = C_2(\alpha) > 0$ , such that if  $\widehat{X} = \{(\theta, X(\theta)); \theta \in J\}$  and  $\varphi^k(\widehat{X}) = \{(\theta, X_k(\theta)); \theta \in J_k\}$  are graphs with  $|X'|, |X'_k| \leq C_1$ , then for all  $z, w \in \varphi^k(\widehat{X})$ ,*

$$\text{dist}_{\widehat{X}}(\varphi^{-k}(z), \varphi^{-k}(w)) \leq C_2 |\partial_\theta(g^k(\theta_k))|^{-1} \text{dist}_{\varphi^k(\widehat{X})}(z, w)$$

for some  $\theta_k \in J$ , where  $\text{dist}_A$  is the distance induced by the metric over the curve  $A$ .

*Proof.* Let us consider the canonical norm in the tangent space, i.e.,  $\|(v_1, v_2)\| = (|v_1|^2 + |v_2|^2)^{\frac{1}{2}}$ , where  $v = (v_1, v_2) \in T_z(\mathbb{T}^1 \times I_0)$ ,  $v_1 \in T_\theta \mathbb{T}^1$ ,  $v_2 \in T_x I_0$  and  $z = (\theta, x)$ .

We denote the tangent vector to the curve  $\widehat{X}$  at the point  $(\theta, X(\theta))$  by  $(v_1(\theta), v_2(\theta))$ . Let us consider  $\theta_z, \theta_w \in J$  such that  $\varphi^k(\theta_z, X(\theta_z)) = z$  and analogously for  $w$ . Then, since  $|v_2(\theta)|/|v_1(\theta)| \leq C_1$ ,

$$\begin{aligned} \text{dist}_{\varphi^k(\widehat{X})}(z, w) &= \int_{\theta_z}^{\theta_w} \|D\varphi^k(\theta, X(\theta))(v_1(\theta), v_2(\theta))\| d\theta \geq \int_{\theta_z}^{\theta_w} |\partial_\theta g^k(\theta)| |v_1(\theta)| d\theta \geq \\ &\geq \frac{1}{(1 + (C_1)^2)^{\frac{1}{2}}} \int_{\theta_z}^{\theta_w} |\partial_\theta g^k(\theta)| (|v_1(\theta)|^2 + |v_2(\theta)|^2)^{\frac{1}{2}} d\theta \\ &\geq \frac{1}{(1 + (C_1)^2)^{\frac{1}{2}}} |\partial_\theta g^k(\theta_k)| \text{dist}_{\widehat{X}}(\varphi^{-k}(z), \varphi^{-k}(w)) \end{aligned}$$

where  $\theta_k$  is such that  $|\partial_\theta g^k(\theta_k)| \leq |\partial_\theta g^k(\theta)|$  for  $\theta \in [\theta_z, \theta_w]$ . This means that we may take  $C_2 = (1 + (C_1)^2)^{\frac{1}{2}}$ .  $\square$

**6.2. Properties of the hyperbolic-like times.** In the case that  $k$  is a hyperbolic time for  $z$ , there is contraction for all the inverse iterates in a certain neighborhood of  $\varphi^k(z)$ . In the case of hyperbolic-like times this property is not necessarily verified. However, it holds the following result

**Proposition 6.3.** *Given  $\sigma > 0$ , there exists  $\delta_1 > 0$  such that for  $z \in M$  with  $r_k(z) \geq \sigma$  for some  $k \in \mathbb{N}$ , there exists a neighborhood  $V_k(z)$  of  $z$  such that  $\varphi^k : V_k(z) \rightarrow B_{\delta_1}(\varphi^k(z))$  is a diffeomorphism with bounded distortion (it depends on  $\sigma$ , but it is independent of  $z$  and  $k$ ).*

*Proof.* Let  $z = (\theta, x) \in \mathbb{T}^1 \times I_0$  for some  $\theta \in \mathbb{T}^1$  and  $x \in I_0$ . Let  $\mathcal{T}_k(z)$  be the maximal interval such that  $\varphi^j(\mathcal{T}_k(z)) \cap \mathcal{C} = \emptyset$  for all  $j < k$  and let  $\mathcal{L}_k(z), \mathcal{R}_k(z)$  be the components of  $\mathcal{T}_k(z) \setminus \{z\}$ . By hypothesis  $|\varphi^k(\mathcal{L}_k(z))| \geq \sigma$  and  $|\varphi^k(\mathcal{R}_k(z))| \geq \sigma$ . Let us consider  $\mathcal{I}_k(z) \subset \mathcal{T}_k(z)$  such that every component of  $\varphi^k(\mathcal{T}_k(z)) \setminus \varphi^k(\mathcal{I}_k(z))$  has length equal to  $\sigma/2$ . In particular, we have that both components of  $\varphi^k(\mathcal{I}_k(z) \setminus \{z\})$  have length greater or equal than  $\sigma/2$ . By definition of  $\varphi$ , we know that the horizontal component of  $\varphi^k(z)$  is  $g^k(\theta)$ . Let us consider  $\eta_1 > 0$  and  $\eta_2 > 0$  such that  $g^k : (\theta - \eta_1, \theta + \eta_2) \rightarrow (g^k(\theta) - \rho', g^k(\theta) + \rho')$  is a diffeomorphism. Here  $\rho'$  is a sufficiently small constant whose value will be made precise in (6.4).

Let  $I_k(z)$  be the projection of  $\mathcal{I}_k(z)$  onto  $I_0$ . Let us consider the set  $B_k(z) = (\theta - \eta_1, \theta + \eta_2) \times I_k(z)$ . For every  $w = (\theta, x_w) \in I_k(z)$ , we denote by  $\mathcal{B}_w$  the line joining the points  $(\theta - \eta_1, x_w)$  and  $(\theta + \eta_2, x_w)$ . We denote by  $\mathcal{B}_w^j$  (for  $j \leq k$ ) the curve given by the image of  $\mathcal{B}_w$  under  $\varphi^j$ , i.e., which satisfies  $\varphi^j(\mathcal{B}_w) = \mathcal{B}_w^j$ . Observe that  $\mathcal{B}_w^0 = \mathcal{B}_w$  for any  $w \in I_k(z)$ .

In the same way we denote by  $w^k$  the image under  $\varphi^k$  of the point  $w = w^0$  and by  $\mathcal{T}^j$  the set  $\varphi^j(\mathcal{T}_k(z))$  (since  $z$  and  $k$  are fixed along the proof, there is no confusion in omitting in the notation the dependence of  $\mathcal{T}^j$  on  $z$  and  $k$ ).

**Claim 6.1.**  $\varphi^k : B_k(z) \rightarrow \varphi^k(B_k(z))$  is a diffeomorphism.

*Proof.* We will use the bounded distortion of the map  $g$ . Namely, there exists  $D > 0$  such that, if we have  $J \subset \mathbb{T}^1$  and  $n \in \mathbb{N}$  for which  $g^n : J \rightarrow g^n(J)$  is a diffeomorphism, then

$$\frac{|\partial_\theta g^n(\theta)|}{|\partial_\theta g^n(\omega)|} \leq D \quad (6.1)$$

for all  $\theta, \omega \in J$ . We claim that  $\mathcal{B}_w^j \cap \mathcal{C} = \emptyset$  for  $j < k$  and for any  $w \in I_k(z)$ .

Recall the constants  $C, C_1, C_2$  and  $D$ , specified in (3.1), Proposition 6.1, Proposition 6.2 and (6.1), respectively. Let us assume that for every  $w \in I_k(z)$ ,  $|\mathcal{B}_w^k| \leq \rho$ , where  $\rho$  satisfies the conditions

$$C_2 \rho < (\sigma/4)(DC)^{-1} \quad \text{and} \quad \rho C_1 < \sigma/4. \quad (6.2)$$

Let us fix  $w \in \mathcal{I}_k(z)$ . First, for all  $j \leq k$ ,  $\mathcal{B}_w^j$  are  $C_1$ -curves (see Definition 6.1 and Proposition 6.1). On the other hand, there exists  $C_2$  such that  $|\mathcal{B}_w^{k-j}| \leq C_2 |\partial_\theta g^j(\theta_j)|^{-1} |\mathcal{B}_w^k|$  for some  $(\theta_j, x_j) \in \mathcal{B}_w^{k-j}$ , where  $|\mathcal{B}|$  denotes the arc length of the curve  $\mathcal{B}$  (see Proposition 6.2).

For  $1 \leq j \leq k$ , let us denote by  $\mathcal{I}_{w,+}^{k-j}$  and  $\mathcal{I}_{w,-}^{k-j}$  the connected components of  $\mathcal{T}^{k-j} \setminus \{w^{k-j}\}$ . By the mean value theorem, we have that  $|\mathcal{I}_{w,+}^{k-j}| \geq (\prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega_j, y_j))|)^{-1} (\sigma/2)$  for some  $(\omega_j, y_j) \in \mathcal{I}_w^{k-j}$ ; and  $|\mathcal{I}_{w,-}^{k-j}| \geq (\prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega'_j, y'_j))|)^{-1} (\sigma/2)$ , for some  $(\omega'_j, y'_j) \in \mathcal{I}_w^{k-j}$ . So, two cases can occur: (i)  $|\mathcal{I}_{w,+}^{k-j}| \leq |\mathcal{I}_{w,-}^{k-j}|$ , or (ii)  $|\mathcal{I}_{w,+}^{k-j}| > |\mathcal{I}_{w,-}^{k-j}|$ .

Let us assume that we have the case (i) (the other case is totally analogous). Then combining (3.1) and (6.1), we have

$$|\partial_\theta g^j(\theta_j)|^{-1} < DC a^j \left( \prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega_j, y_j))| \right)^{-1}.$$

From Proposition 6.2, the last inequality and (6.2), we have for  $1 \leq j \leq k$ ,

$$|\mathcal{B}_w^{k-j}| \leq C_2 |\partial_\theta g^j(\theta_j)|^{-1} \rho < a^j \left( \prod_{i=0}^{j-1} |\partial_x f(\varphi^i(\omega_j, y_j))| \right)^{-1} (\sigma/4) \leq a^j \frac{\text{dist}_{\text{vert}}(w^{k-j}, \mathcal{C})}{2}. \quad (6.3)$$

for  $w \in \mathcal{I}_k(z)$ . This equation, and the condition  $(F_2)$  satisfied by the skew-product, implies that  $\mathcal{B}_w^{k-j} \cap \mathcal{C} = \emptyset$  (for every  $1 \leq j \leq k$ ). Therefore the map  $\varphi^k : B_k \rightarrow \varphi^k(B_k)$  is a local diffeomorphism.

We claim that the map is injective. Indeed, if there exist  $(\theta_1, x_1)$  and  $(\theta_2, x_2)$  in  $B_k$  such that  $\varphi^k(\theta_1, x_1) = \varphi^k(\theta_2, x_2) \in B$ , since in the horizontal direction there is expansion ( $\partial_\theta g > 1$ ), it must be  $\theta_1 = \theta_2$ . Next, by the differentiability of the functions  $f(\theta, \cdot)$ , if  $x_1 \neq x_2$ , there must be at least one point  $(\theta_1, x_w)$  between  $(\theta_1, x_1)$  and  $(\theta_1, x_2)$  and  $j < k$  such that this point is mapped by  $\varphi^j$  in a critical point. But this would imply that  $\mathcal{B}_w^j \cap \mathcal{C} \neq \emptyset$  (for some  $w \in \mathcal{I}_k(z)$ ), which is a contradiction. Hence  $x_1 = x_2$ , which implies that the map  $\varphi^k : B_k \rightarrow \varphi^k(B_k)$  is injective.

Therefore, if  $\rho$  is as in (6.2), Claim 6.1 follows. It just remains to state precisely the value of  $\rho'$ . Given  $\rho$ , we choose  $\rho' < \rho$  maximal such that

$$\begin{aligned} &\text{given } J \subset \mathbb{T}^1 \text{ interval with length } \rho' \text{ and } X : J \rightarrow I_0 \text{ a curve with } |X'| \leq C_1, \\ &\text{the arc length of } \text{graph}(X) \text{ is less or equal than } \rho. \end{aligned} \quad (6.4)$$

where  $C_1$  is the constant given in Proposition 6.1. It finishes the proof of the claim.  $\square$

Let us prove now that the transformation of Claim 6.1 has bounded distortion.

**Claim 6.2.** *There exists  $K_1 = K_1(\sigma) > 0$  such that for  $z_1, z_2 \in \mathcal{I}_k(z) \subset B_k(z)$ ,*

$$\frac{1}{K_1} \leq \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \leq K_1.$$

*Proof.* Let  $z_1$  and  $z_2$  be points in  $\mathcal{I}_k(z)$ , where  $z = (\theta, x)$  for some  $x \in I_0$  and  $\theta \in \mathbb{T}^1$ . We have that  $I_k(z) \subset T_k(z)$  (since  $\mathcal{I}_k(z) \subset \mathcal{T}_k(z)$  and these sets are the corresponding projections onto  $I_0$ ). Recall the notation  $f_\theta^k = f_{g^{k-1}(\theta)} \circ \dots \circ f_{g(\theta)} \circ f_\theta$ , where  $f_\theta(x) = f(\theta, x)$  for  $\theta \in \mathbb{T}^1$  and  $x \in I_0$ . Since  $\varphi^j(\mathcal{T}_k(z)) \cap \mathcal{C} = \emptyset$  for  $j < k$ , we have that  $f_\theta^k : T_k(z) \rightarrow f_\theta^k(T_k(z))$  is a  $C^3$  diffeomorphism. By the way we have chosen  $\mathcal{I}_k(z)$  we know that every component of  $f_\theta^k(T_k(z)) \setminus f_\theta^k(I_k(z))$  has length equal to  $\sigma/2$ . Then there exists  $\kappa > 0$  (depending only on  $\sigma$ ), such that  $f_\theta^k(T_k(z))$  contains a  $\kappa$ -scaled neighborhood of  $f_\theta^k(I_k(z))$

(i.e, both components of  $f_\theta^k(T_k(z)) \setminus f_\theta^k(I_k(z))$  have length  $\geq \kappa|J|$ ). Thus, by Koebe Principle (see [14, Theorem IV.1.2]), there exists  $K_1 = K_1(\kappa) > 0$  such that for  $y_1, y_2 \in I_k(z)$ ,

$$\frac{1}{K_1} \leq \frac{|Df_\theta^k(y_1)|}{|Df_\theta^k(y_2)|} \leq K_1.$$

Now, for  $z_1 = (\theta, y_1) \in I_k(z)$ ,  $|\det D\varphi^k(z_1)| = |\partial_\theta g^k(\theta)| |Df_\theta^k(y_1)|$ . It finishes the proof.  $\square$

**Claim 6.3.** *There exists  $K_2 = K_2(\sigma) > 0$  such that for  $z_1 \in I_k(z)$  and  $z_2$  in the same horizontal leaf  $\mathcal{B}_{z_1}$  of  $z_1$ ,*

$$\frac{1}{K_2} \leq \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \leq K_2.$$

*Proof.* Using the condition  $(F_2)$  satisfied by the skew-product, together with (6.1), we conclude

$$\left| \log \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \right| \leq \log D + B \sum_{j=1}^k \frac{\text{dist}(\varphi^{k-j}(z_1), \varphi^{k-j}(z_2))}{\text{dist}_{\text{vert}}(\varphi^{k-j}(z_1), \mathcal{C})}$$

and by (6.3), we have

$$\left| \log \frac{|\det D\varphi^k(z_1)|}{|\det D\varphi^k(z_2)|} \right| \leq \log D + B \sum_{j=1}^k a^j \leq B' \sum_{j=1}^{\infty} a^j = K'_2.$$

This concludes the proof of the claim.  $\square$

Combining Claim 6.2 and Claim 6.3, we get that  $\varphi^k : B_k(z) \rightarrow \varphi^k(B_k(z))$  has bounded distortion. To finish the proof of Proposition 6.3, it remains to show that  $\varphi^k(B_k(z))$  contains  $B_{\delta_1}(\varphi^k(z))$  for some  $\delta_1 > 0$ .

Recall that  $z = (\theta, x)$ . The image of the horizontal curves of  $B_k(z)$ , i.e.  $\mathcal{B}_w^k$ , are  $C_1$ -curves for all  $w \in I_k(z)$  (see Definition 6.1 and Proposition 6.1). Using this fact and (6.2) we conclude that  $\varphi^k(B_k(z))$  contains the set

$$(g^k(\theta) - \rho', g^k(\theta) + \rho') \times (f_\theta^k(x) - \sigma/4, f_\theta^k(x) + \sigma/4)$$

where  $\rho'$  was defined on (6.4) and it does neither depend on the point  $z$ , nor on the iterate  $k$ . Hence there exists  $\delta_1 > 0$  such that  $B_{\delta_1}(\varphi^k(z)) \subset \varphi^k(B_k(z))$ . Considering  $V_k(z) \subset B_k(z)$  such that  $\varphi^k(V_k(z)) = B_{\delta_1}(\varphi^k(z))$ , Proposition 6.3 follows.  $\square$

**6.3. Neighborhoods associated to hyperbolic-like times.** For every  $\sigma > 0$  and  $i \in \mathbb{N}$ , we denote by  $H_i(\sigma)$  the set of points  $z \in M$  with  $r_i(z) \geq \sigma$ . The following lemma will be very useful in the construction of the absolutely continuous invariant measure for  $\varphi$ .

**Lemma 6.1.** *Given  $\sigma > 0$ , there exists  $\tau = \tau(\sigma) > 0$  such that for every  $i \in \mathbb{N}$  and for any measurable set  $Z$ , there exists a finite set of points  $z_1, \dots, z_N$  in  $H_i(\sigma)$  and neighborhoods  $V'_i(z_1), \dots, V'_i(z_N)$  which are two-by-two disjoint. For every  $k = 1, \dots, N$ ,  $\varphi^i : V'_i(z_k) \rightarrow B_{\delta_1/4}(\varphi^i(z_k))$  is a diffeomorphism with bounded distortion and the union  $W_i = V'_i(z_1) \cup \dots \cup V'_i(z_N)$  satisfies*

$$\text{Leb}(W_i \cap H_i(\sigma) \cap Z) \geq \tau \text{Leb}(H_i(\sigma) \cap Z).$$

**Remark 6.1.** The constant  $\delta_1$  and the distortion bound which appear in this lemma are the same given in Proposition 6.3, which are independent on the point  $z \in M$  and on the iterate  $i \in \mathbb{N}$ .

**Proof.** This is analogous to the proofs of Proposition 3.3 and Lemma 3.4 of [4], using hyperbolic-like times instead of hyperbolic times.

## 7. ABSOLUTELY CONTINUOUS INVARIANT MEASURE

Here we prove Theorem A. In order to do it, we need to control the measure of the points with many (positive density) hyperbolic-like times.

**7.1. Points with infinitely many hyperbolic-like times.** We are going to show that, for some  $\varepsilon > 0$ , the points with many  $\varepsilon$ -hyperbolic-like times are a positive Lebesgue measure set.

Recall that we denote  $\mathbb{T}^1 \times I_0$  by  $M$  and the Lebesgue measure of  $M$  by  $\text{Leb}$ . Given any  $\lambda > 0$ , let  $Z(\lambda)$  be the set of points in  $M$  for which the limit in (2.1) is greater than  $2\lambda$ . Also, for  $n \in \mathbb{N}$ , we define,

$$Z_n(\lambda) = \left\{ z \in Z(\lambda); \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\varphi(\varphi^j(z))^{-1}\|^{-1} > \lambda \right\},$$

and for  $\delta > 0$ ,

$$A_n^M(\delta) = \left\{ z \in M; \frac{1}{n} \sum_{i=1}^n r_i(z) < \delta^2, \quad r_n(z) > 0 \right\}$$

where  $r_i(z) = r_i(\theta, x)$  denotes the function  $r_i(\{f_n\}, x)$  defined on subsection 2.2, considering the sequence  $f_n = f_{g^n(\theta)}$  for  $n \geq 0$ . As we will now see, these sets have relation with the sets defined by equations (4.2) and (4.1).

We denote by  $A_n(\theta, \delta)$  the set  $A_n(\{f_n\}, \delta)$  (defined on (4.1)), and by  $Y_n(\theta, \lambda)$  the set  $Y_n(\{f_n\}, \lambda)$  (defined on (4.2)), with  $f_n = f_{g^n(\theta)}$  for  $n \geq 0$ . Thus, we can conclude that

$$Z_n(\lambda) \subset \cup_{\theta \in T} (\theta \times Y_n(\theta, \lambda)) \quad \text{and} \quad A_n^M(\delta) = \cup_{\theta \in T} (\theta \times A_n(\theta, \delta)). \quad (7.1)$$

For every  $\theta \in \mathbb{T}^1$ ,  $\{f_{g^n(\theta)}\}$  is a  $C^1$ -uniformly equicontinuous and  $C^1$ -uniformly bounded sequence of smooth maps. It also holds that  $p = \sup \# \mathcal{C}_{g^n(\theta)} < \infty$ . Thus, we are in the context of Lemma 4.1. Moreover, for fixed  $\lambda > 0$ , the constant  $\delta$  given by Lemma 4.1 does not depend on  $\theta$ , i.e., the constant  $\delta$  is the same for any sequence  $\{f_{g^n(\theta)}\}$ . This happens because the modulus of continuity (2.3), the uniform bound  $\Gamma$  in (2.4) and the uniform bound  $p$  for the number of critical points, are the same for any sequence  $\{f_{g^n(\theta)}\}$  (varying  $\theta \in \mathbb{T}^1$ ). The last is true since  $\varphi$  is  $C^3$  and  $(F_1)$  holds.

**Proposition 7.1.** *In the conditions of Theorem A, given  $\lambda > 0$ , there exist  $\varepsilon = \varepsilon(\lambda) > 0$  such that*

$$\text{Leb} \left( \left\{ z \in Z(\lambda); \sum_{i=1}^n r_i(z) \geq 2\varepsilon n, \text{ for all } n \geq n_0 \right\} \right) \geq \text{Leb}(\cap_{n \geq n_0} Z_n(\lambda)) / 2.$$

for  $n_0$  big enough. Moreover, for Lebesgue almost every  $z \in Z(\lambda)$ ,  $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n r_i(z) \geq 2\varepsilon$ .

*Proof.* For  $\lambda, \delta > 0$  and every  $N \in \mathbb{N}$ ,

$$\begin{aligned} \int_{\mathbb{T}^1} \int_{I_0} \chi_{\{\cap_{n=N}^\infty A_n^M(\delta) \cap Z_n(\lambda)\}}(\theta, x) dm_{I_0}(x) dm_{\mathbb{T}^1}(\theta) &\geq \int_{\mathbb{T}^1} \int_{I_0} \chi_{\{\cap_{n=N}^\infty Z_n(\lambda)\}}(\theta, x) dm_{I_0}(x) dm_{\mathbb{T}^1}(\theta) - \\ &\quad - \int_{\mathbb{T}^1} \int_{I_0} \chi_{\{\cup_{n=N}^\infty A_n^M(\delta) \cap Z_n(\lambda)\}}(\theta, x) dm_{I_0}(x) dm_{\mathbb{T}^1}(\theta). \end{aligned}$$

where  $m_{I_0}$  and  $m_{\mathbb{T}^1}$  denote the Lebesgue measure on  $I_0$  and  $\mathbb{T}^1$ . On the other hand, by Lemma 4.1, there exists  $\delta > 0$  such that for every  $\theta \in \mathbb{T}^1$ ,

$$m_{I_0} \left( \bigcup_{n=N}^\infty A_n(\theta, \delta) \cap Y_n(\theta, \lambda) \right) \rightarrow 0,$$



when  $N \rightarrow \infty$ . This together with (7.1) yield,

$$\int_{\mathbb{T}^1} \int_{I_0} \chi_{\{\cup_{n=N}^{\infty} A_n^M(\delta) \cap Z_n(\lambda)\}}(\theta, x) dm_{I_0}(x) dm_{\mathbb{T}^1}(\theta) \leq \int_{\mathbb{T}^1} \int_{I_0} \chi_{\{\cup_{n=N}^{\infty} A_n(\theta, \delta) \cap Y_n(\theta, \delta)\}}(x) dm_{I_0}(x) dm_{\mathbb{T}^1}(\theta) \rightarrow 0$$

when  $N \rightarrow \infty$ . Considering  $\varepsilon$  such that  $2\varepsilon < \delta^2$ , the proposition follows.  $\square$

**7.2. Positive density of the hyperbolic-like times.** We prove that for every point  $z$  such that  $\sum_{i=1}^n r_i(z) \geq 2\varepsilon n$ , (for some  $\varepsilon > 0$ ), the density of hyperbolic-like times is uniformly positive.

Recall that for every  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , we denote by  $H_n(\varepsilon)$  the set of points  $z \in M$  with  $r_n(z) \geq \varepsilon$ .

**Lemma 7.1.** *Given  $\varepsilon > 0$ , there exists  $\zeta = \zeta(\varepsilon) > 0$  such that*

$$\frac{\#\{1 \leq i \leq n; z \in H_i(\varepsilon)\}}{n} \geq \zeta$$

for any  $z$  such that  $\sum_{i=1}^n r_i(z) \geq 2\varepsilon n$ .

*Proof.* Considering  $c_2 = 2\varepsilon$  and  $c_1 = \varepsilon$ , applying the Pliss lemma (see [17]), there are  $q \geq \zeta n$  and  $0 < n_1 < \dots < n_q \leq n$  such that

$$\sum_{j=k+1}^{n_i} r_j(z) \geq \varepsilon(n_i - k) \quad \text{for every } 0 \leq k < n_i, \text{ and } i = 1, \dots, q.$$

Observe that  $\zeta$  does not depend on  $z$  neither on  $n$ . Hence, for any  $z$  as in the statement of the lemma, there exist  $0 < n_1 < \dots < n_q \leq n$  such that  $r_{n_i}(z) \geq \varepsilon$  ( $1 \leq i \leq q$ ) and  $q/n \geq \zeta$ .  $\square$

**7.3. Construction of the measure.** We consider the sequence

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \varphi_*^i \text{Leb}$$

of averages of forward iterates of Lebesgue measure on  $M$ . The main idea is to decompose  $\mu_n$  (for every  $n$ ) as a sum of two measures,  $\nu_n$  and  $\eta_n$ , such that  $\nu_n$  is uniformly absolutely continuous and has total mass bounded away from zero. The measure  $\nu_n$  will be the part of  $\mu_n$  carried on balls of radius  $\delta_1$  around points  $\varphi^i(z)$ , where  $z$  is a point which has  $1 \leq i \leq n$  as  $\varepsilon$ -hyperbolic-like time.

Let us fix  $\lambda > 0$  such that  $\text{Leb}(Z(\lambda)) > 0$ . Let us consider the corresponding  $\varepsilon = \varepsilon(\lambda) > 0$  from Proposition 7.1. Let  $W_i$  be the set given by Lemma 6.1 for  $\sigma = \varepsilon$ . We consider the measures

$$\nu_n = \frac{1}{n} \sum_{i=1}^n \varphi_*^i \text{Leb}_{W_i}$$

and  $\eta_n = \mu_n - \nu_n$ , where  $\text{Leb}_X$  denotes the restriction of the Lebesgue measure to  $X$ .

**Proposition 7.2.** *The measures  $\nu_n$  are uniformly absolutely continuous and give positive (bounded away from zero) weight to  $Z(\lambda)$ , for all large  $n$ .*

*Proof.* By Proposition 6.3, the measures  $\varphi_*^i \text{Leb}_{V_i(z)}$  are absolutely continuous and the densities are uniformly bounded from above. It also holds for the measures  $\varphi_*^i \text{Leb}_{W_i}$ , since  $W_i$  is a disjoint union of sets  $V_i'$ s. Therefore,  $\nu_n$  are absolutely continuous and the densities are uniformly bounded from above. It just remains to prove the claim about  $Z(\lambda)$ . By Lemma 6.1, there exists  $\tau = \tau(\varepsilon)$  such that

$$\nu_n(Z(\lambda)) \geq \tau \frac{1}{n} \sum_{i=1}^n \text{Leb}(H_i(\varepsilon) \cap Z(\lambda)).$$

So, it suffices to control the right side of the last expression. For this, let us consider the measure  $\pi_n$  in  $\{1, 2, \dots, n\}$  defined by  $\pi_n(B) = \#(B)/n$ , for every subset  $B$ . Using Fubini's theorem, we have

$$\frac{1}{n} \sum_{i=1}^n \text{Leb}(H_i(\varepsilon) \cap Z(\lambda)) = \int \int_{Z(\lambda)} \chi(z, i) d\text{Leb}(z) d\pi_n(i) = \int_{Z(\lambda)} \int \chi(z, i) d\pi_n(i) d\text{Leb}(z)$$

where  $\chi(z, i) = 1$  if  $z \in H_i(\varepsilon)$  and  $\chi(z, i) = 0$  otherwise. By Lemma 7.1, it holds  $\int \chi(z, i) d\pi_n(i) \geq \zeta$  if  $z$  is such that  $\sum_{i=1}^n r_i(z) \geq 2\varepsilon n$ . Hence

$$\frac{1}{n} \sum_{i=1}^n \text{Leb}(H_i(\varepsilon) \cap Z(\lambda)) \geq \zeta \text{Leb}\left(\left\{z \in Z(\lambda); \sum_{i=1}^n r_i(z) \geq 2\varepsilon n\right\}\right).$$

In this way, we conclude using Proposition 7.1 that the weight of  $Z(\lambda)$  for the measure  $\nu_n$  is bounded away from zero, for  $n$  big enough.  $\square$

The limit of any convergent subsequence of  $\{\nu_n\}_n$  is an absolutely continuous measure. It just remains to prove that we can find our measure in such a way that it is invariant. Let us choose  $\{n_k\}_k$  such that  $\mu_{n_k}$ ,  $\nu_{n_k}$  and  $\eta_{n_k}$  converge to  $\mu$ ,  $\nu$  and  $\eta$ , respectively. We can decompose  $\eta = \eta^{ac} + \eta^s$  as the sum of an absolutely continuous measure  $\eta^{ac}$  and a singular measure  $\eta^s$  (with respect to Lebesgue measure). Then,  $\mu = (\nu + \eta^{ac}) + \eta^s$  gives one decomposition of  $\mu$  as sum of one absolutely continuous and one singular measure. Since the push forward under  $\varphi$  preserves the class of absolutely continuous measures and  $\mu$  is invariant,  $\mu = \varphi_*\mu = \varphi_*(\nu + \eta^{ac}) + \varphi_*\eta^s$  gives another decomposition of  $\mu$  as sum of one absolutely continuous and one singular measure. By the uniqueness of the decomposition we must have  $\varphi_*(\nu + \eta^{ac}) = \nu + \eta^{ac}$ . Hence,  $\nu + \eta^{ac}$  is a non-zero absolutely continuous invariant measure for  $\varphi$ .

**7.4. Ergodicity and finite number of measures.** To finish the proof of Theorem A, it remains to prove the ergodicity of the absolutely continuous invariant measure and the finiteness claim in the statement of the theorem. Fixed  $\lambda > 0$ , we consider the constant  $\varepsilon > 0$  given on Proposition 7.1. Recall that for  $\sigma = \varepsilon$ , we denote by  $V_k(z)$  (for  $k \in \mathbb{N}$ ,  $z \in M$ ) the neighborhood constructed on Proposition 6.3: it is mapped diffeomorphically onto the ball of radius  $\delta_1 > 0$  around  $\varphi^k(z)$  by  $\varphi^k$ .

**Lemma 7.2.** *Let  $\lambda > 0$  and  $\varepsilon = \varepsilon(\lambda)$  be as in Proposition 7.1. Let us consider  $G_0 \subset M$  an open set. Then for any  $z \in Z(\lambda) \cap G_0$ ,  $V_k(z) \subset G_0$  whenever  $z \in H_k(\varepsilon)$  and  $k$  is big enough.*

*Proof.* In Proposition 6.3 we fixed the constant  $\rho'$  according to (6.4) and we constructed the neighborhood  $V_k(z)$ . This neighborhood is such that  $V_k(z) \subset B_k(z) = (\theta - \eta_1, \theta - \eta_2) \times I_k(z)$ , where: (i)  $g^k : (\theta - \eta_1, \theta + \eta_2) \rightarrow (g^k(\theta) - \rho', g^k(\theta) + \rho')$  is a diffeomorphism; (ii)  $I_k(z) \subset T_k(z)$  and  $f_\theta^k$  is a diffeomorphism restricted to  $T_k(z)$ . To conclude the proof, it is enough to show that  $\eta_1$ ,  $\eta_2$  and  $|I_k(z)|$  goes to zero when  $k$  goes to infinity. The claim about  $\eta_1$  and  $\eta_2$  follows from the uniform expansion of  $g$ . Since  $z \in Z_k(\lambda)$  for  $k$  big enough, the bounded distortion on  $f_\theta^k : I_k(z) \rightarrow f_\theta^k(I_k(z))$  (see the proof of Claim 6.2) implies that  $|I_k(z)|$  goes to zero.  $\square$

**Lemma 7.3.** *For any positively invariant set  $G \subset Z(\lambda)$  there exists some disk  $\Delta$  with radius  $\delta_1/4$  such that  $\text{Leb}(\Delta \setminus G) = 0$ .*

*Proof.* The proof is analogous to the proof of Lemma 5.6 of [4]. We make use of  $\varepsilon(\lambda)$ -hyperbolic-like times instead of  $(\sigma, \delta)$ -hyperbolic times. Thus, the only difference is the reason why the neighborhoods  $V_k(z)$  decrease with  $k$ . In our case, this is given by Lemma 7.2.  $\square$

*End of proof of Theorem A.* At the end of subsection 7.3, we construct an absolutely continuous invariant measure  $\nu_0 := \nu + \eta^{ac}$  with  $\nu_0(Z(\lambda)) > 0$ . Since  $Z(\lambda)$  is positively invariant, we can suppose

that  $\nu_0(Z(\lambda)) = 1$ . On the other hand, by Lemma 7.3, each invariant set on  $Z(\lambda)$  with positive  $\nu_0$ -measure has full Lebesgue measure in some disk with fixed radius. Since the manifold is compact, there can be only finitely many disjoint invariant sets on  $Z(\lambda)$  with positive  $\nu_0$ -measure. Hence  $\nu_0$  can be decomposed as a sum of ergodic measures. Namely,  $\nu_0 = \sum_{i=1}^l \nu_0(D_i)\nu_i$ , where  $D_1, \dots, D_l$  are disjoint invariant sets with positive measure and  $\nu_i$  is the normalized restriction of  $\nu_0$  to  $D_i$ . The measures  $\nu_i$  ( $1 \leq i \leq l$ ) are ergodic absolutely continuous probabilities. Therefore, they are SRB measures.

If  $Z_1 = Z(\lambda) \setminus \bigcup_{i=1}^s B_i$  (where  $B_i$  denotes the basin of the measure  $\mu_i$ ) has positive Lebesgue measure, then we can repeat the arguments in this section with  $Z_1$  in the place of  $Z(\lambda)$ . Thus we construct new absolutely continuous invariant ergodic measures. Repeating this procedure, we find absolutely continuous invariant ergodic measures such that almost every point in  $Z(\lambda)$  is in the basin of one of these measures. The number of measures is finite since the basins are invariant sets and Lemma 7.3 holds. It finishes the proof of Theorem A.  $\square$

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